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NUMERICAL SOLUTION OF FLOOD PREDICTION AND RIVER REGULATION PROBLEMS

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Report III

Results of the Numerical Prediction of the 1945 and 1948 Floods in the Ohio River, of the 1947 Flood Through the Junction of the Ohio and Mississippi Rivers, and of the Floods of 1950 and 1948 Through Kentucky Reservoir.

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Errata

page	
4	Fourth line from bottom: replace "surve" by "curve"
12	Middle of page: replace "they are about $1\frac{1}{2}$ feet" by "they are also quite small until the last week when they reach a maximum of $1\frac{1}{2}$ feet."
33	Eighth line from top: replace "functions" by "function"
39	Second line from bottom: should read "modification"
51	Eighth line from top: insert "to" after the word "over"
54	Middle of page: delete "over the banks" and replace by "into the river from tributaries and local drainage between Wheeling and Cincinnati"
59	Ninth line from footnote: replace "solvable" by "solved"
62	Middle of page: delete "(pumps)"
65	Sixth line from bottom: replace "outlines" by "outlined"
66	Eighth line from top: replace "if" by "If"
67	Fourth line from bottom: replace "tributaries" by "turbines"

Acknowledgment

The work presented in this report, as well as in two earlier reports, was carried out under a contract with the Corps of Engineers of the U. S. Army. In addition, the constant and whole-hearted cooperation of B. R. Gilcrest, E. A. Lawler, and F. U. Druml of the Ohio River Division of the Corps of Engineers were available in collecting the basic data for the Ohio River and interpreting it. Excellent basic data for the calculations referring to Kentucky Reservoir were supplied by the Tennessee Valley Authority.

The initial stages in planning and coding the numerical computations on the UNIVAC for the Ohio River and for its junction with the Mississippi were carried out by Dr. Milton E. Rose. The completion of the calculations for the Ohio River, and the planning and coding of the problems concerning Kentucky Reservoir were carried out by Harold Shulman. Mrs. Halina Montvila aided in checking by hand the coding for the UNIVAC and also prepared, in collaboration with Frank Druml, the code for converting rainfall data in the Upper Ohio Valley into overbank inflows. A great deal of hand computing and drawing of graphs was done by Mrs. Montvila, Miss Miriam Weissner and Mr. Albert Brown.

A good many of the computations were carried out on the UNIVAC belonging to the Army Map Service of the Corps of Engineers in Washington, D. C. We owe special thanks to this organization for the efficient and cooperative manner in which they made their machine available to us. We also wish to thank the Univac installation of the Atomic Energy Commission at New York University for permitting us to fit a good deal of our work into their crowded schedule.

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NUMERICAL SOLUTION OF FLOOD PREDICTION AND RIVER REGULATION PROBLEMS

REPORT III

Results of the Numerical Prediction of the 1945 and 1948 Floods in the Ohio River, of the 1947 Flood through the Junction of the Ohio and Mississippi Rivers, and of the Floods of 1950 and 1948 through Kentucky Reservoir.

§1. Introduction and Summary

In two previous reports (Reports I and II from the New York University Institute of Mathematical Sciences, Nos. 200 (1953) and 205 (1954)) the basic general theory for the numerical analysis of flood wave problems in rivers was developed and applied to simplified models of the Ohio River and of its junction with the Mississippi River. It was found that the numerical treatment of flood problems seemed feasible from the standpoint of the amount of calculating time needed for a digital computer of the type of the UNIVAC and thus it seemed likely that problems for actual rivers could also be successfully solved.

The present report has as its purpose the application of the methods described in the first two reports to the flood wave problems indicated in the title of this report. It might be said at the outset that the results of these calculations show the numerical method to be feasible and practical.

The data for the flood in the Ohio were taken for the case of the big flood of 1945 and predictions were made numerically for periods of two weeks or more for the 375 mile long stretch of the Ohio extending from Wheeling, West Virginia to Cincinnati, Ohio. For the 1948 flood in the Ohio, 6 day runs were made during open river conditions. For the flood through the junction of the Ohio and Mississippi Rivers the data for the 1947 flood were used and predictions were made in

all three branches for distances of roughly 40 miles from the junction along each branch and for periods up to 16 days. In Kentucky Reservoir, which extends from Kentucky Dam near the mouth of the Tennessee River 184 miles upstream to Pickwick Dam, flood predictions were made for the flood of 1950 for a period of 21 days, and further calculations for the 1948 flood for 7 1/4 days were made. In each case the state of the river or river system was taken from the observed flood at a certain time t = 0; for subsequent times the inflow from tributaries and the local runoff in the main river valley were taken from the actual records. The differential equations which characterize the flow of the river were then integrated numerically with the use of the UNIVAC digital computer in order to obtain the river stages at future times. The time required to perform the calculations for the Ohio River on the UNIVAC is at present one minute for a one hour prediction, and less than one-half minute for a one hour prediction in the other two cases. (We estimate that the same calculations on the IBM 704 could be done in less than 1/15 of the time required by the UNIVAC.) The flood stages determined in this way were then compared with the actual records of the flood.

The fact that such flood wave problems in rivers can be solved in this way is, of course, a matter of considerable practical importance from various points of view. Once the basic data for a river, river system, or reservoir have been prepared and coded for a calculating machine, it becomes possible to solve all sorts of problems quickly and inexpensively: for example, the effect on a flood wave of damming a tributary, the relative merits of various schemes for serial operation of dams, or the preparation of tables showing the influence of changing the operating conditions in large reservoirs, are all problems which can be successfully attacked numerically. These observations refer to the use of the UNIVAC computer. The situation would be even more favorable if newer and faster computing machines such as the

IBM 704 were to be used. It is, of course, also of interest to compare the method of numerical computation with the method of using hydraulic models; the authors give their ideas and opinions on this point in section 8 of this report.

Once it becomes clear that the integration of the differential equations characterizing flows and wave motions in rivers can be done accurately without an inordinate amount of expensive calculating machine time - and this was already indicated by what was done in Report II mentioned above - it follows that the success of the method for actual rivers hinges on the possibility of obtaining accurate data from which to calculate coefficients and initial and boundary data for the differential equations.

There are four such coefficients (cf. equations (2.1) of the following section): the cross section area A and the breadth B of the river, the resistance coefficient G, and the inflow q from tributaries plus the local inflow from the main valley. The first two quantities are purely geometrical in character and could in principle be determined from topographic surveys (as is, in fact, done when hydraulic models are built). The resistance coefficient G is determined empirically from records of past floods; records of discharge and stage along the river are needed for this purpose. The quantities A, B, and G are all functions of the location x along the river, and of the stage H. The inflow q is assumed to be a known function of location x and time t. In the problems to be treated here, which were set up to test the method of numerical calculation, this quantity was obtained from the records of an actual flood; in a prediction problem it would be obtained from knowledge of the gaged flows from the larger tributaries plus estimates of the local runoff from rainfall. The quantities A, B, and G are all fixed functions of distance along the river and stage for a given river, river system, or reservoir. The inflow q will, of course, vary from one flood to another as will the initial

and boundary data. In the two earlier reports it was seen that a knowledge of the state of the river at some initial time (usually taken to be $t = 0$) is necessary in order to determine the flow uniquely. This means that both stage and velocity must be known initially; as a rule, the initial velocity is determined by converting discharge data into velocity data by using cross section areas, and the discharges in turn are fixed from actual measurements or from rating curves (in most cases, the latter). Since only finite lengths of any system are in question, it is necessary also to prescribe boundary conditions at the upstream and downstream ends of the system; for tranquil, or subcritical, flow (the only case dealt with here, since only rivers with relatively low flow velocities were in question) it is necessary to prescribe one condition at each endpoint, which might be stage, or velocity, or a relation between the two. In general, the discharge (or, what comes to the same thing, velocity) at the upstream end should be prescribed, since that would seem the natural condition in practice, while at the downstream end of an open river it seems reasonable to use an average rating curve to furnish a relation between stage and discharge; in any case, a condition there must be prescribed which has the effect of simulating the influence of the river below the endpoint. In the upper Ohio River, however, we have prescribed stage at both upstream and downstream endpoints (since our basic object was really to check the general feasibility of the numerical method). In the problem concerning the junction of the Ohio and Mississippi Rivers we have carried out the solution first by prescribing stage at the two upper endpoints (i.e., upstream in the Ohio and upper Mississippi), and afterwards by using discharge data at the upper ends. In both cases a rating surve at the lower end in the Mississippi was used to furnish a boundary condition. Good results were obtained in both cases. (As will be explained later, it is also necessary to impose appropriate continuity conditions at

the junction itself.) In Kentucky Reservoir, which is closed at both ends by dams, the boundary conditions at both ends were formulated in terms of discharge.

Once the coefficients A, B, G, and q are known, and initial and boundary data are given, the differential equations have, for all future times, uniquely determined solutions for the stage H and velocity v as functions of distance x along the river and time t.

As was explained in the earlier reports, the differential equations are integrated approximately by the method of finite differences, which yields values for the stage H and velocity v at a discrete set of points forming a rectangular net in the x,t-plane. In the Ohio River a net with intervals $2\Delta x = 10$ miles in the x-direction along the river, and $\Delta t = 9$ minutes in the t-direction was used, and nets having approximately these spacings were also used in the other cases. This means that the coefficients and the initial data need be known only at the net points, and these quantities should consequently represent averages over 10-mile stretches, but their values are taken at 5-mile intervals since a staggered net was used. (Our preliminary work on simplified models of the Ohio River, as discussed in Report II, indicated that these interval sizes are sufficiently small to yield accurate enough approximations to the solutions of the differential equations. However, the simplified models were such that the differential equations had constant coefficients; in the actual cases the coefficients are variable - so much so that 10-mile intervals are just barely small enough to yield a reasonable approximation. More will be said on this point later on.) The determination of these coefficients as averages over 10-mile intervals was a laborious*, difficult, and crucial part of our task. It is of such importance that the method of doing it will be described

* The data characterizing the geometrical and dynamical parameters for 375 miles of the Ohio River required the tabulation of 1100 constants, for example.

in detail in §3 below. It should be emphasized that this heavy task requires close cooperation with engineers familiar with the data; we were particularly fortunate in having the cooperation of the engineers from the Ohio River Division whose names are given in the acknowledgment.

Although we describe in §3 the methods used by us in converting the basic data for a river into data suitable for our method of numerical calculation, it is nevertheless of interest to indicate here in summary fashion how it was done. Consider first the resistance coefficient G, which depends physically upon the roughness and also upon the nature of the cross section of the river bed. It must be obtained from records of past floods, and it would be very convenient for this purpose to have simultaneous records of flood stages and discharges at points closely spaced along the river. Unfortunately, measurements of discharge are as a rule available only at the ends of rather wide intervals, called reaches - of the order of 60-96 miles in length, even in the Ohio River, for which the data are more extensive than for most rivers in the United States. Thus an initial estimate for the resistance coefficient is obtained as an average over a distance considerably greater even than the interval size of 10 miles upon which our finite difference scheme is based*; a linear interpolation from the midpoints of successive reaches was used to fix the values of G at intermediate net points. As was mentioned above, the cross section area A and the breadth B of the river are geometric quantities which could be obtained from topographic maps. However, such a procedure is extremely laborious and time consuming, and since another equally important empirical element, the resistance coefficient, is known only as an average over each of the reaches, it seemed

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* We had anticipated difficulties because of this fact, but, fortunately they did not materialize in all of the cases. For example, in the junction problem and in Kentucky Reservoir, the first estimates from the basic data for this coefficient were changed very little subsequently. However, quite considerable changes from the first estimates were necessary in the Ohio River.

reasonable to make use of an average cross section area over each reach also. In fact, one of the important aspects of the results to be reported here is that it is indeed possible to make accurate flood predictions by using average cross section areas in an appropriate way. Roughly speaking, this was done in the Ohio River by analyzing data from past floods in such a way as to obtain the storage volume in each reach as a function of the stage, from which an average cross section area is at once determined. At intermediate net points the area A was fixed by linear interpolations from the midpoint of a reach to the midpoint of the adjacent one. In Kentucky Reservoir, however, we were supplied directly with storage volumes (obtained from topographic maps) for intervals of about 10 miles in length. The breadth B is, in principle, the derivative $\frac{dA}{dH}$ of the cross section area with respect to stage. It was to be anticipated that this quantity would be somewhat sensitive, and this proved to be the case; how reasonable average values for it were computed from the data is perhaps best left to the detailed description in §3. Thus the quantities A, B, and G are computed as numerically tabulated functions of stage at each of the net points along the river. However, in order to save number storage capacity in the UNIVAC, these quantities were fitted to empirical curves (quadratic, cubic, and hyperbolic curves in different cases) with a few parameters: the details are of some importance, and they also are discussed in §3. The quantity $q(x,t)$ which yields the inflow data, is of course taken directly from the records; any gaged flows from tributaries were put in at the nearest interval, while the ungaged local drainage for a given reach was distributed uniformly over the intervals in that reach.

It would naturally be too much to expect that a given flood would be accurately reproduced by numerical integration on the first trial. In order to improve the numerical results it is in general necessary to adjust the resistance coefficient and the average cross section areas and breadths,

which are obtained initially as averages in an appropriate sense over those past floods for which data are available, to bring this about. This is, in fact, what is also done when making model studies, in which, however, all adjustments are made through varying the roughness of the model. In effect, the observed flood is used as a means to correct first estimates of the physical parameters. It had been anticipated that very extensive changes in roughness coefficients would be necessary (as it is with model studies) in order to reproduce a given flood accurately. Actually, it turned out that the first estimates of the roughness coefficient were quite good in two of the three cases, but that the average cross section areas and breadths were in need of revisions, particularly in some reaches of the Ohio River. In the junction problem, and in Kentucky Reservoir, no really extensive revisions of the initial estimates for either the resistance or the area and breadth coefficients were necessary*, while both of these quantities had to be varied considerably in order to reproduce the 1945 flood in the Ohio River with reasonable accuracy. What this means is that the type of basic data available in two of the three cases treated by us sufficed to fix accurately the geometrical and dynamical parameters which govern the flows, and that consequently the differential equations in these cases are adequate and correct formulations of the basic laws which determine the flows uniquely. In the case of the Ohio River, it would be necessary to make still more revisions in the coefficients then we have had opportunity to make so far before it would be sure that the differential equations mirror accurately the characteristics of the river. In

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* In Kentucky Reservoir, however, the mesh width of 10 miles was too large for the finite difference scheme that worked well in the other two cases. We therefore used a more accurate finite difference scheme in the calculations for Kentucky Reservoir. In fact, 10 miles is, we now feel, just on the borderline of what is reasonable as an interval size for the simple and straightforward ways in which we approximated derivatives by difference quotients.

particular, the use of average cross section areas obtained from past flood records by balancing flows to obtain storage volumes is probably not accurate enough, and ought to be replaced by averages from topographic maps.

In §4 the results of flood predictions for the 1945 and 1948 floods in the 375 mile stretch of the Ohio River between Wheeling and Cincinnati are described in detail. In Fig. 1.1 we give a graph showing a typical result; the graph shows the observed and the calculated stages for the 1945 flood at Maysville for 13 1/2 days. The time required on the UNIVAC to make the computations was 6 3/4 hours. As one sees, the agreement is generally good. The error at the crest of the flood was 0.1 foot, and the maximum error (late on March 6) was 1.6 feet. Upon going back to the basic data, and looking at a map of the drainage areas, it was observed that the ungaged inflow in this reach which was quite high for a short time on March 6, was mainly introduced not far above Maysville; on the other hand, in making our calculations (as we have mentioned above) the ungaged inflow was distributed over the whole reach, and it is thus not surprising that the calculated hydrograph is smoother than the one actually observed. Later on, the curves came together again.

Our process of calculation in all three of the problems studied involves, at bottom, the replacement of the actual river in all of its complexity by a model in which average properties (average areas, resistance factors, etc.) over distances of varying length come into play. It turns out in all three of our cases that the calculated stages given by such models agree well with the actual stages all along the river, but that the discharges obtained by taking the product of the calculated velocity and the cross section area of the model at a given point may disagree widely with the local discharge as observed in the actual river - at gaging stations at the ends of reaches, especially. The reason for this discrepancy is that the gaging stations are invariably placed

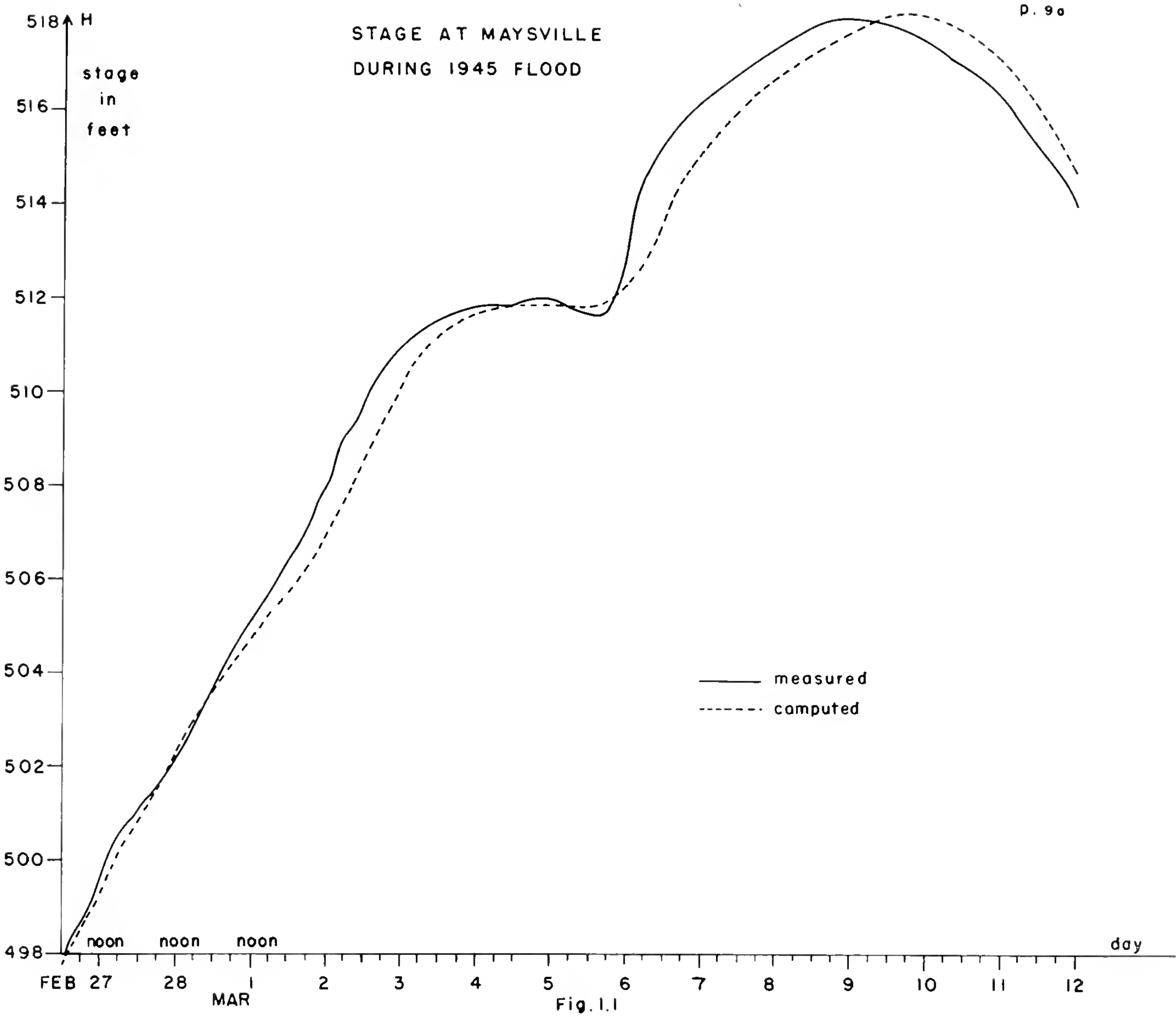
STAGE AT MAYSVILLE
DURING 1945 FLOOD
H
stage
in
feet
518
516
514
512
510
508
506
504
502
500
498
measured
computed
noon
noon
noon
day
FEB 27
28
MAR
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 1.1

at narrow portions of the river, and thus the local areas and resistance coefficients at such points differ widely from the averages used by us. It is nevertheless possible to obtain correct local discharges by making an easy supplementary calculation which has the effect of passing back to the actual river from the model. The method of doing so is explained in §4 in connection with the problem of the upper Ohio River; the same method applies in the other cases, but will not be repeated in the discussion of these cases.

It has already been stated above that quite extensive changes had to be made in the initial estimates for all of the coefficients - resistance coefficient G, area A, and breadth B - in the case of the Ohio River in order to reproduce, as was done successfully, the observed stages of the 1945 flood. The fact that this was necessary is already a strong indication that the basic data were inadequate (or, perhaps, not used in the best way by us) to characterize the Ohio River accurately*. In fact, when the differential equations were used subsequently to predict the stages in the 1948 flood, the results were not accurate at some (though not all) of the gaging stations. (This was particularly true in the region about Huntington. It has perhaps some significance to add that we are told by the engineers of the Ohio River Division that they also have difficulty in getting their calculations to check in this vicinity.) In §4 the details of the results for the 1948 flood verification, and a discussion of the possible causes for errors and of possible ways to overcome them, is given. In a way, it was unfortunate that we began our work with the Ohio River, since we are now convinced that the problem presented by the Ohio River is by far the most difficult of the three treated by us, (for reasons which will be given in later sections), and that it should be studied more carefully

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* There are discrepancies in the storage volumes of as much as 20 % when determined by different methods, and one could have legitimate doubts with respect to the accuracy of the inflows between Wheeling and Cincinnati, which are a very large part of the total flow.

than our resources in time and funds permitted: with our present experience we feel that we could attack the problem in better ways.

In §5 the calculations for the 1947 flood through the junction of the Ohio and Mississippi Rivers are described and analyzed. We reproduce here in Figs. 1.2(a) and (b) graphs showing observed and computed stages at Cairo, the junction, and at Hickman in the Mississippi River below the junction. Approximately 40 miles in each of the three branches is involved, and discharge was prescribed at the upper ends, while a rating curve relating stage and discharge was used as a boundary condition at Hickman. The computation time on the UNIVAC for the flow over a period of 20 days was about 3 1/2 hours. As one sees, the observed and calculated stages are in very good agreement, with a maximum error of about 6 inches at Cairo and about 1 foot at Hickman. It is to be seen that there is a uniform bias at both Hickman and Cairo in the sense that the observed stages are lower on the rising part of the flood and higher on the falling part than the calculated stages. This is doubtlessly the result of using a (single-valued) simple rating curve at Hickman as a basis for fixing the relation between stage and discharge that was used as a boundary condition. Rating curves which depend on the water surface slope as parameter should perhaps have been used, since that is what is actually observed. Had this been done, the correction would have been in such a direction as to decrease the discrepancy between observed and calculated stages since the actual rating curve relation would, for rising stages, furnish a lower stage for a given discharge than that used in the calculations, and just the reverse for falling stages. One observes that the bias at Cairo is still noticeable, though less in value than at Hickman. Only minor changes were made in the cross section areas and resistance coefficients from the values computed initially from the basic data.

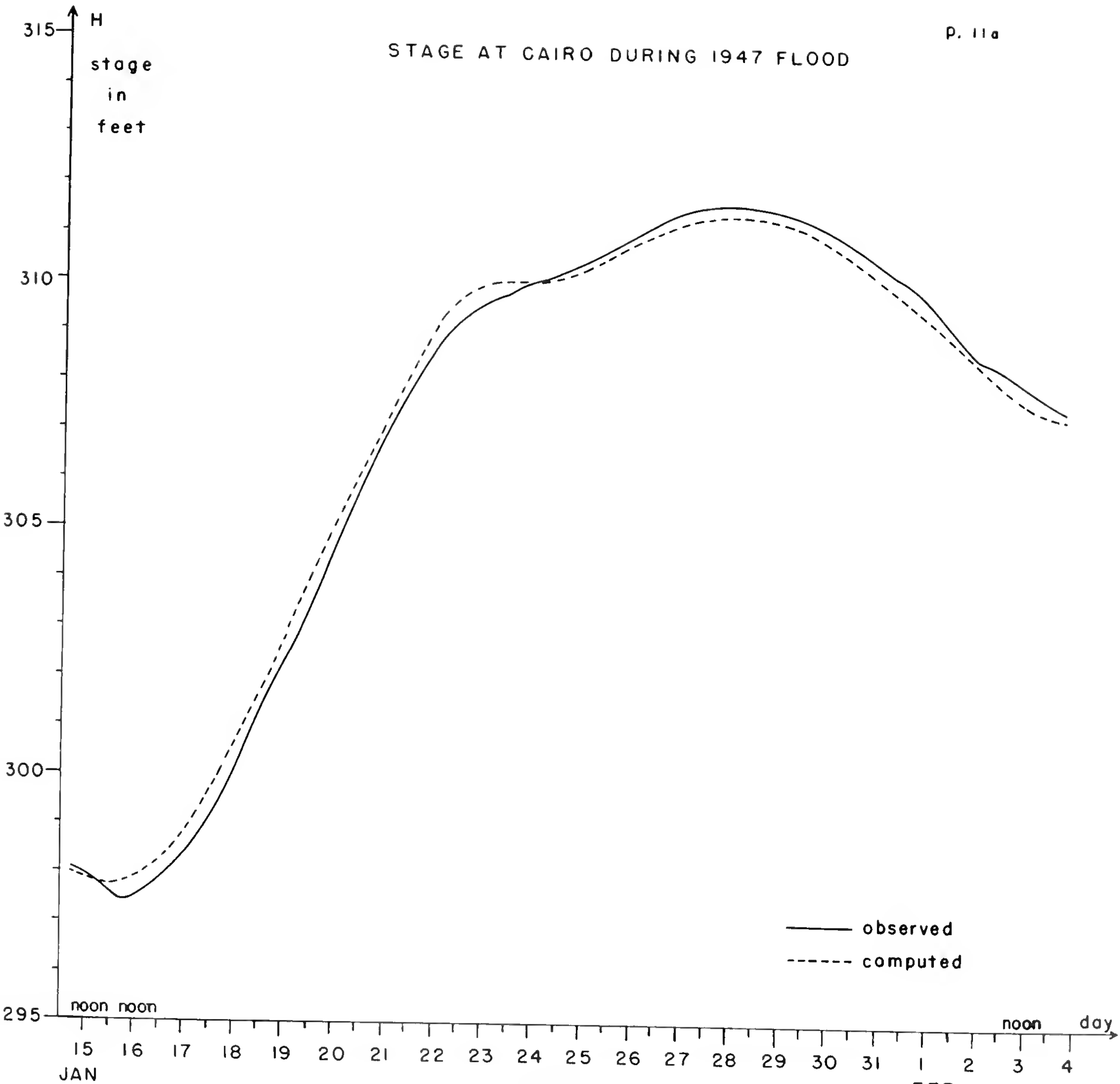
STAGE AT CAIRO DURING 1947 FLOOD
H
stage
in
feet
315
310
305
300
295
noon noon
noon
day
15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 1 2 3 4
JAN
FEB
observed
computed

Fig 1.2a

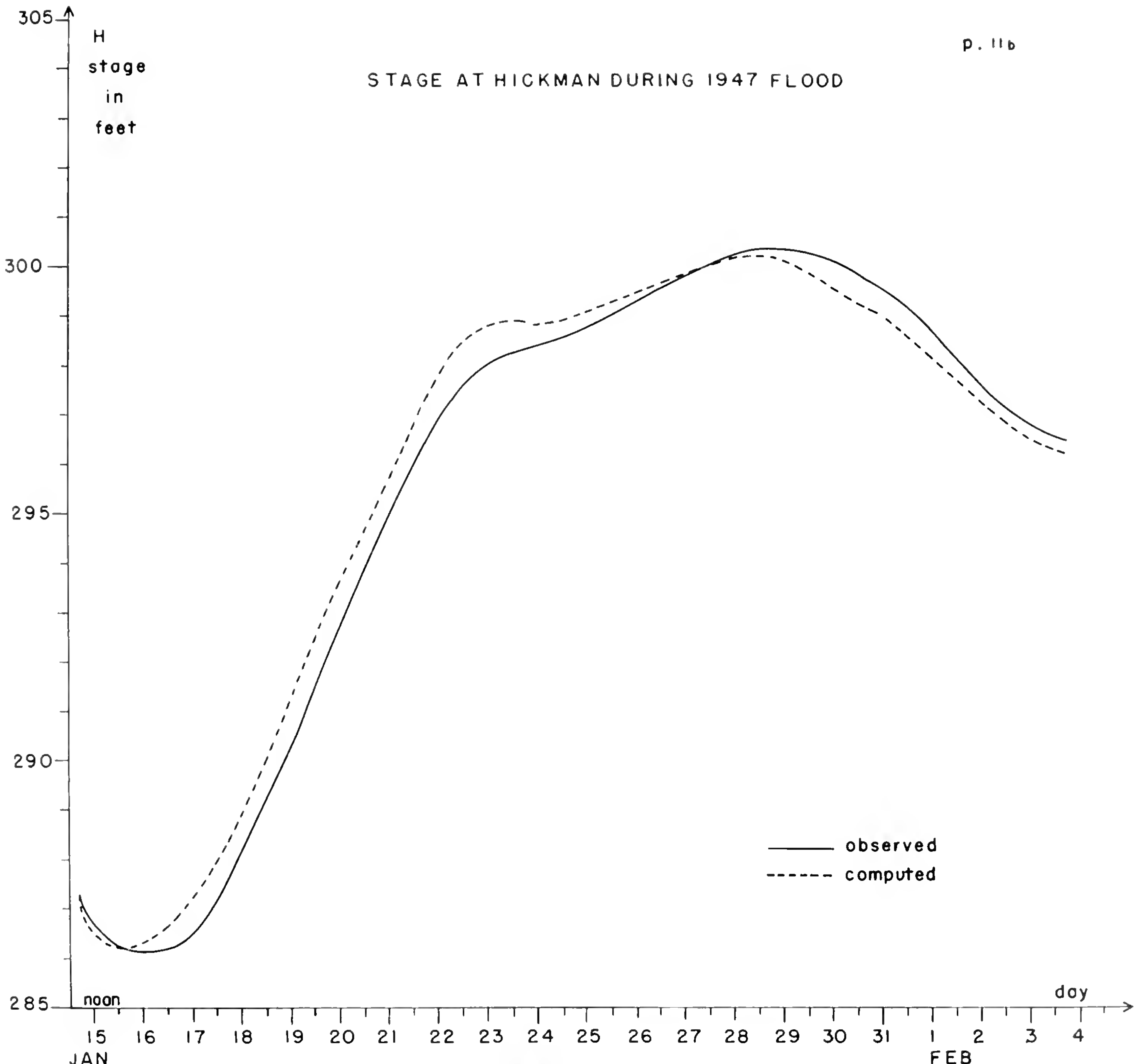
STAGE AT HICKMAN DURING 1947 FLOOD
H
stage
in
feet
305
300
295
290
285
noon
day
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
1
2
3
4
JAN
FEB
observed
computed

Fig. 1.2b

In §6 the calculations for the 1948 and 1950 floods through Kentucky Reservoir are described and analyzed. As was mentioned above, the reservoir is 184 miles long and extends from Pickwick Dam at the upstream end to Kentucky Dam downstream. Discharge data were used to obtain boundary conditions at both ends of the reservoir. In Fig. 1.3 results of the calculations for the 1950 flood are shown. Stages were calculated for a 21-day period beginning Jan. 4, and a little less than 4 hours of UNIVAC time was needed for the computation. Stages at Pickwick and Kentucky Dams* (the ends of the reservoir), and at Perryville (close to the middle of the reservoir) are shown. The agreement between observed and calculated stages is seen to be very good. At Pickwick Dam and Perryville the errors are of the order of a few inches, while at Kentucky Reservoir they are about 1 1/2 feet. One sees, moreover, that even the minor variations in the observed stages are reproduced faithfully by the calculated stages. These results again were obtained without extensive changes in resistance, area, and breadth coefficients after their determination from the original data. A revision and refinement of the finite difference scheme was, however, necessary (as was mentioned above), because the mesh width of 10 miles was too large in comparison with the rapid variations in cross section area and width with location along the reservoir. The coefficients used for the 1950 flood were then used in the differential equations to calculate the progress of the 1948 flood through the reservoir. In this case the flood is reproduced quite accurately. The progress of the 1948 flood was calculated for 7 1/4 days only, since the stages were then higher than those of the 1950 flood (hence it has no meaning to speak then of a verification), and also the increase in stage was so rapid in some places as to make our finite

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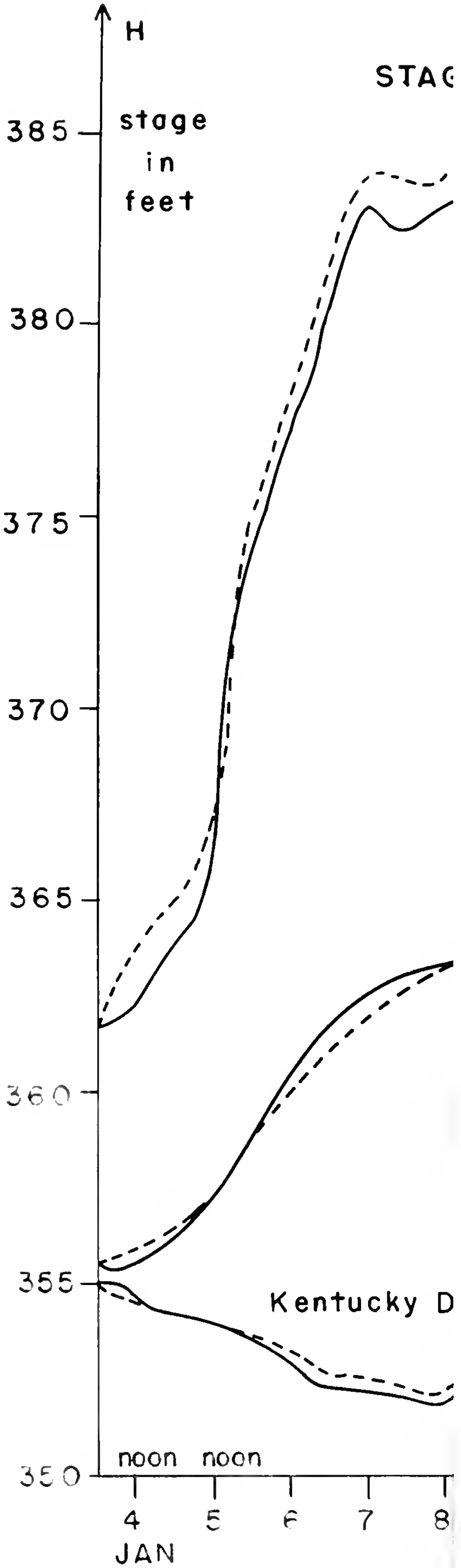
* Since discharges only are prescribed at these points, the stages are determined as part of the solution of the differential equations.


H
STAG
stage
in
feet
385
380
375
370
365
360
355
350
Kentucky D
noon
noon
4
5
6
7
8
JAN

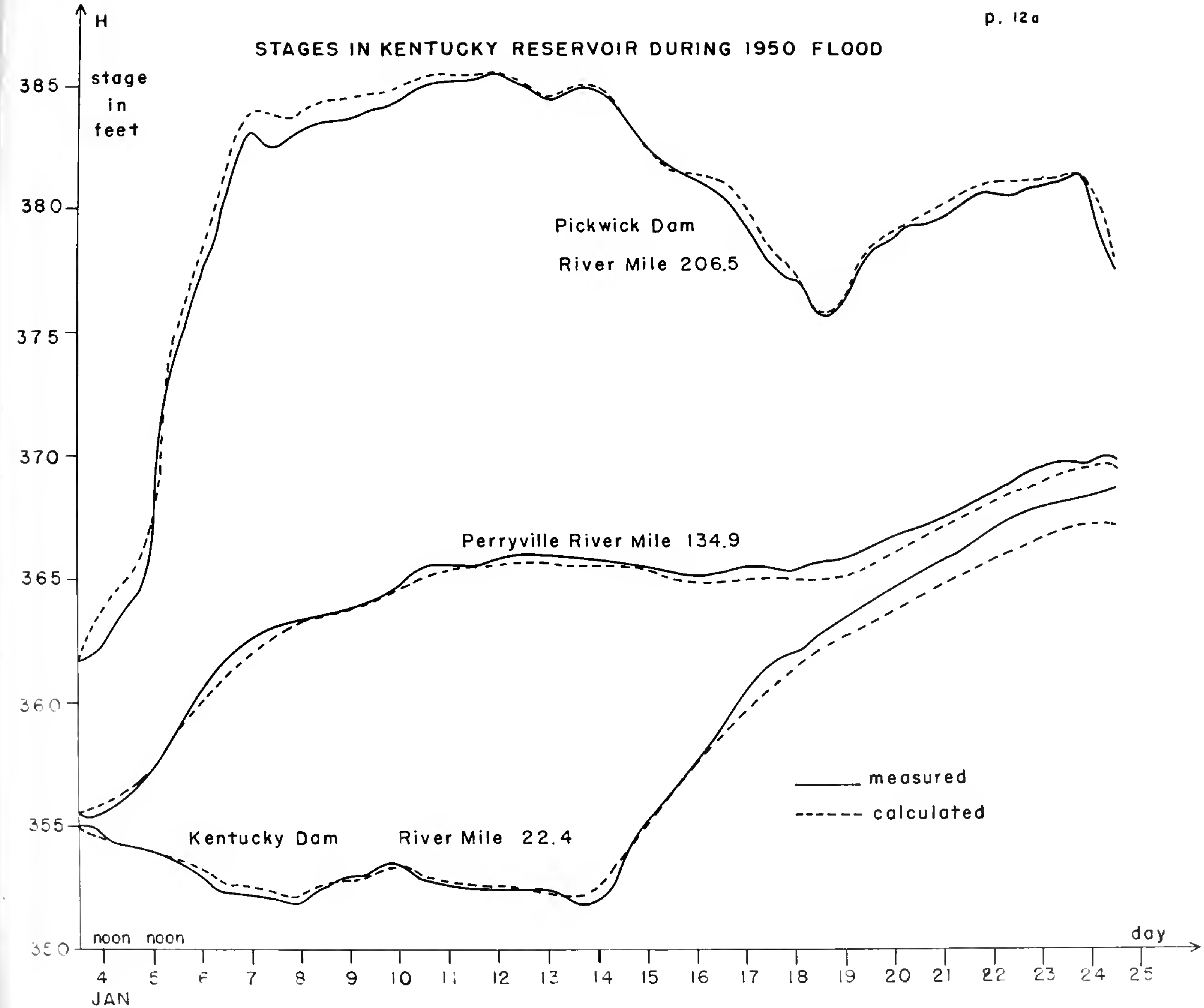

H
STAGES IN KENTUCKY RESERVOIR DURING 1950 FLOOD
stage in feet
385
380
375
370
365
360
355
350
Pickwick Dam
River Mile 206.5
Perryville River Mile 134.9
Kentucky Dam
River Mile 22.4
measured
calculated
noon noon
day
4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
JAN

Fig. 1.3

difference approximations too crude: either a finer net should be used, or a more refined scheme of approximation should be devised.

Summing up, we observe that the numerical method proved to be very successful for the junction problem and for Kentucky Reservoir, and partially so for the Ohio River. The character of the results in the first two cases - in particular, the fact that even minor variations in the observed stages were reproduced with only minor changes and adjustments in the coefficients - convinces us of the feasibility and practicality of the numerical method. The authors feel sure that the Ohio River problem can also be solved by numerical methods as accurately as the basic data permits.

In §7 we set down some suggestions, based on our experience, for modifying the numerical methods. In §8 we discuss the relation between the numerical methods used by us for flood and river regulation problems and the method of studying floods by means of models of rivers. The views and interpretations presented there are those of the authors, and do not reflect necessarily the views of any others who have been associated with us in this enterprise. In §9 we describe a method by which rainfall data can be converted into runoff data to obtain the ungaged local inflows in a form suitable for machine computation; this was carried out for the Ohio River.

Finally, the instructions to the UNIVAC needed to solve any of the problems discussed here will be made available, though in a restricted number of copies because of their bulk.

§2. Outline of the Numerical Methods Used for Solving Flow Problems in Rivers

The basic equations governing the flow in a river are

$$
(2.1) \qquad
\begin{aligned}
&BH_t + (Av)_x = q \\
&v_t + vv_x + gH_x = -Gv|v| - \frac{q}{A} v \quad .
\end{aligned}
$$

In these equations H denotes the elevation of the water surface above sea level, v the velocity of the flow in the river, G the resistance coefficient, A the cross section area, B the width of the river at the water surface, q the volume of inflow over the river banks and from tributaries per unit length and time. These equations correspond to equations (2.3) and (2.11) of Report I. However, the term A_t in (2.3) of the earlier report is now written in the obviously equivalent form BH_t, while the coefficient of the resistance term in (2.11) of Report I has been simplified by writing it as a function G(x,H) which is to be empirically determined. The derivation of these equations, which are in any case well known, has been given in Report I. The notation here is slightly different from that of Report I; in particular, it might be noted that the slope S of the river bottom does not appear explicitly in the second of equations (2.1). S is contained in the term involving H_x since $H_x = y_x - S$, where y is the depth of the river. (It should be noted that H is considered positive upward, but the slope S is taken to be positive although the downstream direction is taken as the positive x-direction.)

The solution of any concrete flood wave problem requires the determination of H and v as functions of the location x along the river and of the time t. As was explained in Reports I and II these quantities can be obtained by integrating the equations (2.1) (once the coefficients A,B,G are known)

provided that the initial values for H and v are known at some time, say at $t = 0$, and if in addition the inflow over the banks and from tributaries is given as a function of time: that is, the quantity q in the first of equations (2.1) should be supposed known as a function of x and t. It is of course possible to deal only with finite lengths of a given river or river system and as a consequence boundary conditions are needed at the end points. In the problem of predicting a flood through the junction of the Ohio and Mississippi Rivers, for example, boundary data were applied as follows: The discharge was assumed known above the junction at Metropolis in the Ohio and at Thebes in the Upper Mississippi. At the downstream end of the Mississippi at Hickman it was assumed that the relation between stage and discharge was known. In addition, at Cairo, the junction of the three branches, it was necessary to fulfill transition conditions: these took the form of requirements that the stage in all three branches was the same and that the inflows from the upper branches just balanced the outflow into the Lower Mississippi.

As was described in the preceding reports, numerical integration of the equations (2.1) with the given initial and boundary conditions is to be performed by using the method of finite differences. However, before writing down the requisite formulas involving finite differences there is some point in rewriting equations (2.1) in the so-called characteristic form for two reasons: first of all it is important to determine the slopes of the characteristics in order to fix a maximum safe interval for the time increment Δt, and secondly, this form of the equation is appropriate for use in computations at boundary points, namely at upstream and downstream ends of the river stretch to be investigated.

In order to put the equations (2.1) in characteristic form the second equation is multiplied by $\pm\sqrt{\frac{BA}{g}}$ and added to the first equation thus yielding the system

$$(2.2)\qquad \begin{aligned} &(Av)_x + BH_t - q + \sqrt{\frac{BA}{g}}\left\{v_t + vv_x + gH_x + Gv|v| + \frac{qv}{A}\right\} = 0 \\ &(Av)_x + BH_t - q - \sqrt{\frac{BA}{g}}\left\{v_t + vv_x + gH_x + Gv|v| + \frac{qv}{A}\right\} = 0 . \end{aligned}$$

These can in turn be rewritten in the following form

$$(2.3)\qquad B\left\{\frac{d}{d\sigma}H\right\} \pm \frac{c}{g}\frac{d}{d\sigma}v = q \mp \frac{Bc}{g}\left\{Gv|v| + \frac{qv}{A}\right\} - vA_x,$$

where $\frac{d}{d\sigma} = \frac{\partial}{\partial t} + [v \pm c]\frac{\partial}{\partial x}$, and $c = \sqrt{\frac{gA}{B}}$. The quantity c represents the speed of propagation of small disturbances (in the case of a rectangular channel $c = \sqrt{gy}$, with y the depth of the stream) and $\frac{d}{d\sigma}$ denotes differentiation in one of the two characteristic directions. That is

$$(2.4)\qquad \frac{dx}{dt} = v \pm c$$

are the characteristic directions in the x,t plane in which the derivatives of H and v are taken in equation (2.3), and the solution curves of these first order ordinary differential equations are called the characteristics corresponding to a given solution H(x,t), v(x,t) of (2.1).

The method of finite differences is based on the determination of approximate solutions of the differential equations in a discrete net of points in an x,t-plane. There are various procedures which can be used to determine such approximate solutions. In general, use was made of a staggered rectangular net as indicated in figure 2,1.*

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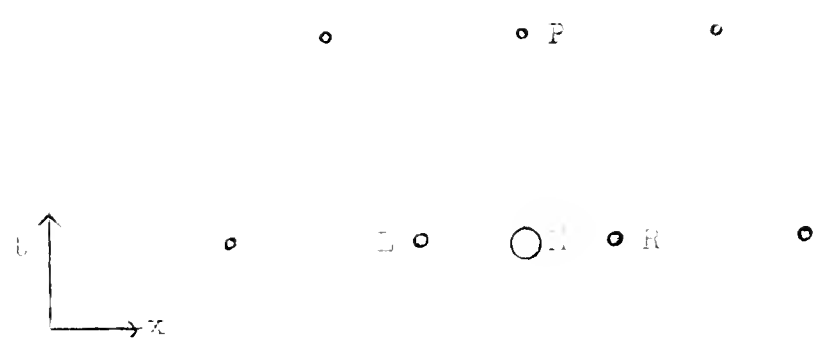
*The reasons for choosing such a net are discussed in the earlier reports.



F i g u r e   2.1

Staggered net point lattice in interior

The general idea of the method of finite differences is to advance the solution step-wise in time intervals of length Δt. Suppose, for example, the values of H and v have already been obtained in a certain horizontal row of net points, say in a row containing the points L and R of figure 2.1. The method of advancing the solution to the next row is then as follows: Consider the point M midway between the points L and R. The values of H and v at this point are defined as the following averages

$$(2.5) \qquad H_M = \tfrac{1}{2}(H_R + H_L) \ , \quad v_M = \tfrac{1}{2}(v_R + v_L) \ .$$

The derivatives of H and v at M are approximated by difference quotients in an obvious way; and these approximations to the derivatives at M are then inserted for the corresponding derivatives in equations (2.1). The result is a pair of algebraic equations which can be solved to yield approximate values for v and H at point P. The results are

$$v_p = \frac{1}{2}[v_R+v_L] - \frac{\Delta t}{2\Delta x}\left\{\frac{v_R^2 - v_L^2}{2} + g(H_R - H_L)\right\}$$

$$- \frac{\Delta t}{2}[G_R v_R^2 + G_L v_L^2] - \frac{\Delta t}{2}\left[\frac{v_R}{A_R} + \frac{v_L}{A_L}\right] q_{RL}$$

(2.6)

$$H_P = \frac{1}{2}[H_R + H_L] - \frac{\Delta t}{(B_R+B_L)\Delta x}[A_R v_R - A_L v_L] + 2\Delta t \left[\frac{q_{RL}}{B_R + B_L}\right]$$

The criterion for convergence of the finite difference scheme as Δx and Δt tend to zero is that P should always lie within a triangle formed by the segment LR and the two characteristics issuing from L and R of slope $v \pm c$, given by equation (2.4).

In order to compute values of stage or discharge for net points at either the upstream or downstream end, it is necessary to have given one physical condition, such as stage or discharge or a relation between stage and discharge such as a rating curve. The physical condition is then used with one of the differential equations in characteristic form to determine the boundary values of H and v. For a boundary point on the upstream side the appropriate characteristic equation* (i.e. the <u>second</u> equation of (2.2)) when put in finite difference form is used first with respect to a point K, not in the staggered lattice, and then for the point N which is in the staggered lattice, as indicated in Fig. 2.2 below. (we recall that the positive x - direction is taken downstream).

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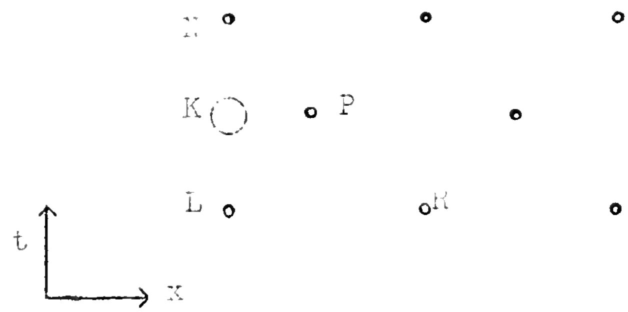
*See Report I, p.30 for an explanation of this type of procedure.



F i g u r e 2.2

Net point scheme at left boundary

That is, we use the given physical condition together with equations (2.7) and (2.8) below to determine H_N and v_N in terms of already known quantities:

$$B_L\left[\frac{H_K-H_L}{\Delta t}\right] + \frac{A_R v_R - A_L v_L}{2\Delta x} - q_{RL}$$

(2.7)

$$-\sqrt{\frac{B_L A_L}{g}}\left[\frac{v_K-v_L}{\Delta t} + \frac{v_R^2-v_L^2}{4\Delta x} + g\left(\frac{H_R-H_L}{2\Delta x}\right) + \frac{q_{RL}\ v_L}{A_L} + G_L v_L^2\right] = 0$$

$$B_K\left[\frac{H_N-H_K}{\Delta t}\right] + \frac{A_P v_P - A_K v_K}{\Delta x} - q_{KP}$$

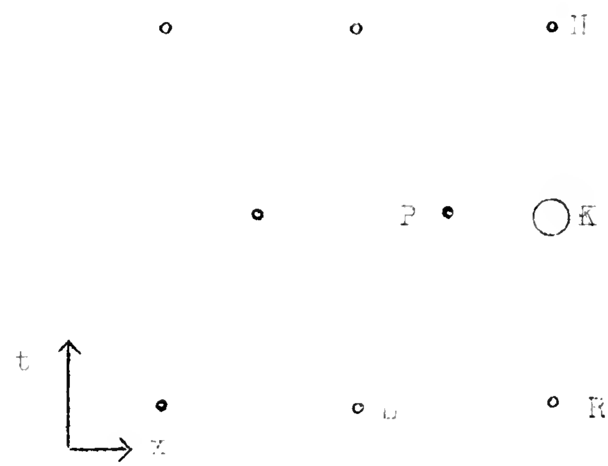
(2.8)

$$-\sqrt{\frac{B_K A_K}{g}}\left[\frac{v_N-v_K}{\Delta t} + \frac{v_P^2-v_K^2}{2\Delta x} + g\left(\frac{H_P-H_K}{\Delta x}\right) + \frac{q_{KP} v_K}{A_K} + G_K v_K^2\right] = 0.$$

By q_{RL} and q_{KP} we mean the inflows over the appropriate segments at the appropriate time.

At the downstream end a similar pair of equations (2.9) and (2.10) is set up, as indicated in Reports I and II, with

the additional point K, as indicated in Figure 2.3



F i g u r e 2.3

$$
(2.9)\quad
\begin{aligned}
&B_R\left[\frac{H_K - H_R}{\Delta t}\right] + \frac{A_R v_R - A_L v_L}{2\Delta x} - q_{LR} \\
&+ \sqrt{\frac{B_R A_R}{g}}\left[\frac{v_K - v_R}{\Delta t} + \frac{v_R^2 - v_L^2}{4\Delta x} + g\left(\frac{H_R - H_L}{2\Delta x}\right) + \frac{q_{LR} v_R}{A_R} + G_R v_R^2\right] = 0 ,
\end{aligned}
$$

$$
(2.10)\quad
\begin{aligned}
&B_K\left[\frac{H_N - H_K}{\Delta t}\right] + \frac{A_K v_K - A_P v_P}{\Delta x} - q_{PK} \\
&+ \sqrt{\frac{B_K A_K}{g}}\left[\frac{v_N - v_K}{\Delta t} + \frac{v_K^2 - v_P^2}{2\Delta x} + g\left(\frac{H_K - H_P}{\Delta x}\right) + \frac{q_{PK} v_K}{A_K} + G_K v_K^2\right] = 0.
\end{aligned}
$$

In the case of the junction problem, the values of H and v belonging to the net point at the junction are computed from the conditions that the stages in each of the three branches are equal and that the volume of water which flows in from the upstream branches leaves through the downstream branch, together with equation (2.9) and (2.10) for each of the upstream branches and equations (2.7) and (2.8) for the downstream branch (i.e. the lower Mississippi). The

various conditions to be satisfied at the junction can be reduced to a single implicit equation for the stage at the junction in the form

$$H = F(H) \tag{2.11}$$

where F is an explicitly known function of the unknown junction stage. Equation (2.11) is solved by a process of iteration, in which the first guess at the solution is taken to be the stage H at the previous time. This value is inserted in F(H) to obtain a corrected value of H, etc.; this process converges rapidly.

In the Kentucky Reservoir problem it was at once noticed that the finite difference methods discussed above, which were satisfactory for the Upper Ohio and for the junction of the Ohio and the Mississippi, failed to give correct results. The reason for this failure is the rapid variation in the coefficients of the differential equation with the distance. In other words, the essential quantities A and B, the area and width of the reservoir, varied much too rapidly* in relation to the mesh width $2\Delta x$ of 10 miles, to permit a good approximation to the solution of the differential equations; in fact, the numerical results were so wild as to indicate strong divergence. In addition, it seems likely that the difficulty was aggravated by our use of a staggered net: if the variations in the coefficients happened to be roughly periodic with a period of ten miles, i.e. with a period equal to the mesh width, it is clear

---

*It should be said that we were in a position to know these variations in the quantities A and B accurately since the Tenn. Valley Authority had furnished us with excellent data in the form of averages for these quantities over 10-mile intervals. The changes of width and area are sometimes quite abrupt - they vary by a factor of two or three in adjacent segments.

that the shifting of the net back and forth at each time step Δt = 9 min. could easily result in the building up of a systematic error. In fact, this does happen.

The obvious way to overcome the difficulty would be to decrease the mesh width in the x-direction from ten miles to not more than, say, three miles. But since the time step Δt would have to be decreased in the same proportion the calculating time on a digital computer would be increased by a factor of 9 at least. This would make the use of the Univac somewhat impractical, but it would not necessarily matter if a faster machine such as the IBM 704 were to be used. We, however, were using the Univac, and hence found it necessary to devise a different way to overcome the difficulty. Since the difficulty was felt to arise because of the shifting back and forth of the net in the x-direction at the time intervals Δt, it was thought that it might be eliminated by approximating time derivatives through the use of values at the time $t - \Delta t$ in addition to those at the time t in order to advance them to the time $t + \Delta t$ - while in all previous schemes values at time t only were used as a basis for advancing the solution to the time $t + \Delta t$. This more complicated method of approximating time derivatives (which also complicates the coding for the Univac) turned out to be much more accurate, and it led to a satisfactory solution of the problem.*

The new scheme for approximating derivatives, which we refer to as a centered difference scheme, is described

---

*We have been led to wonder whether some of the difficulties encountered in the Upper Ohio might not also be partially overcome by using this technique. More will be said on this point later.

on the basis of Fig. 2.4. The essential new factor is that the values of v and H

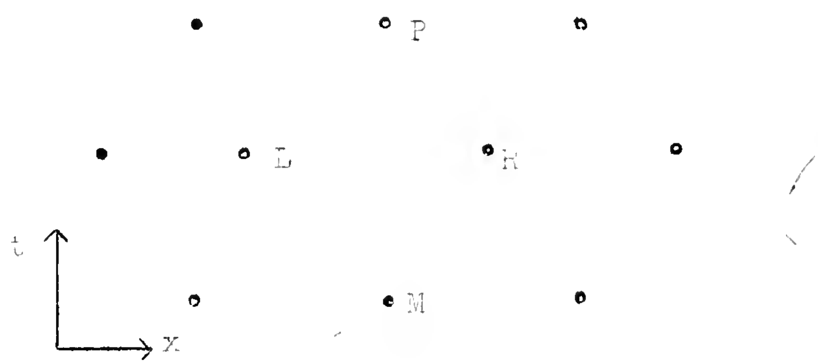


Figure 2.4

Centered Net Point Scheme

are advanced to the point P by making use of the known values of these quantities not only at points L and R but also at point M as well. In fact, the time derivatives are calculated using values at P and M, while x-derivatives are computed in the same way as in the staggered scheme described above, which requires using values at L and R only. The result is the following difference equations as approximations to the differential equations (2.11):

$$\frac{v_P - v_M}{2\Delta t} + \frac{(v_R^2 - v_L^2)}{2(2\Delta x)} + g \cdot \frac{(H_R - H_L)}{2\Delta x} + G_M \cdot v_M^2 + \frac{q_{LR} v_M}{A_M} = 0$$

(2.11) and

$$B_M \frac{(H_P - H_M)}{2\Delta t} + \frac{(A_R v_R - A_L v_L)}{2\Delta x} - q_{LR} = 0$$

These equations can be solved for v_P and H_P as follows to yield the approximate solution at point P determined

from the known values at earlier times:

$$v_P = v_M - 2\Delta t \left\{ \frac{v_R^2 - v_L^2}{4\Delta x} + g\,\frac{(H_R - H_L)}{2\Delta x} + G_M v_M^2 + \frac{q_{LR} \cdot v_M}{A_M} \right\},$$

(2.12)

$$H_P = H_M - \frac{2\Delta t}{B_M} \left\{ \frac{(A_R v_R - A_L v_L)}{2\Delta x} - q_{LR} \right\}.$$

§3. Methods of Obtaining Coefficients and Initial and Boundary Data for the Differential Equations from Empirical Data. Fixing of the Maximum Permissible Time Step.

In the differential equations (2.1) we must fix the cross section area A, the width of the river B, and the resistance coefficient G all as functions of the distance x along the river and of the stage H (actually only at points used in the finite difference scheme). These coefficients are fixed once for all for a given river no matter what special problemsare to be solved. The remaining coefficient, characterizing the flow over the banks and from the tributaries as fixed by the quantity q, will differ from problem to problem depending on the known or estimated run-off from rainfall and the inflows from tributaries. It is supposed given in any specific flood wave problem.

The method of obtaining the coefficients from the basic data depends upon the river itself and also upon the type of information available. Actually several different methods have been used by us: at least two for the Ohio River, another for the junction of the Ohio and Mississippi, and still another for the Kentucky Reservoir.

a) Determination of A and B as functions of x and H:

We begin by discussing the determination of the cross section area A. This coefficient, as was indicated above, must be known as a function of H at every net point. Ideal for this purpose would be actual cross section areas from topographical surveys at points close enough so that accurate average cross sections over ten mile intervals could be obtained. (As we have said repeatedly our net points have been chosen ten miles apart in the staggered scheme of net points). Of course what is wanted is an average cross section area for ten mile stretches centered at each of the net points. For the case of the Kentucky

Reservoir this information was furnished directly since storage volumes "for level pool" were provided. That is, the volume V of the water in ten mile stretches was given as a function of the stage H; the average cross section area was then simply obtained by dividing the storage volume by the length of the segment. Such detailed data were not available for the Ohio River between Wheeling and Cincinnati, and other means of determining average cross section areas had to be devised. It would have been possible and probably would have given better results to do it from topographic maps; this is what is done in constructing a model of a river. However, such a procedure is extremely laborious and time consuming and it was thought preferable to find out to what extent such refinements are necessary, or rather to find out whether rougher and quicker methods of determining cross sections would not be just as satisfactory.

We proceed to describe the methods used by us for determining the cross section area A for the 375 mile stretch of the Ohio River from Wheeling to Cincinnati. There are main gaging stations at Wheeling, St. Marys, Pomeroy, Huntington, Maysville, and Cincinnati, as indicated schematically in Fig. 3.1. At these stations, which are from sixty to ninety miles or so apart, the cross section areas are known as a function of stage. One might think it reasonable to take the cross section areas at the measuring stations as representative of the channel. Unfortunately, these measuring stations are located in general at points where the river is relatively narrow (which is natural, of course, since this facilitates measurement of the discharge) so that these areas do not represent a mean cross section over a reach. In fact a comparison of the mean cross section areas for a reach as obtained below with the areas at gaging stations shows the latter to be as little as half the average area over a reach.

REACHES IN THE OHIO RIVER

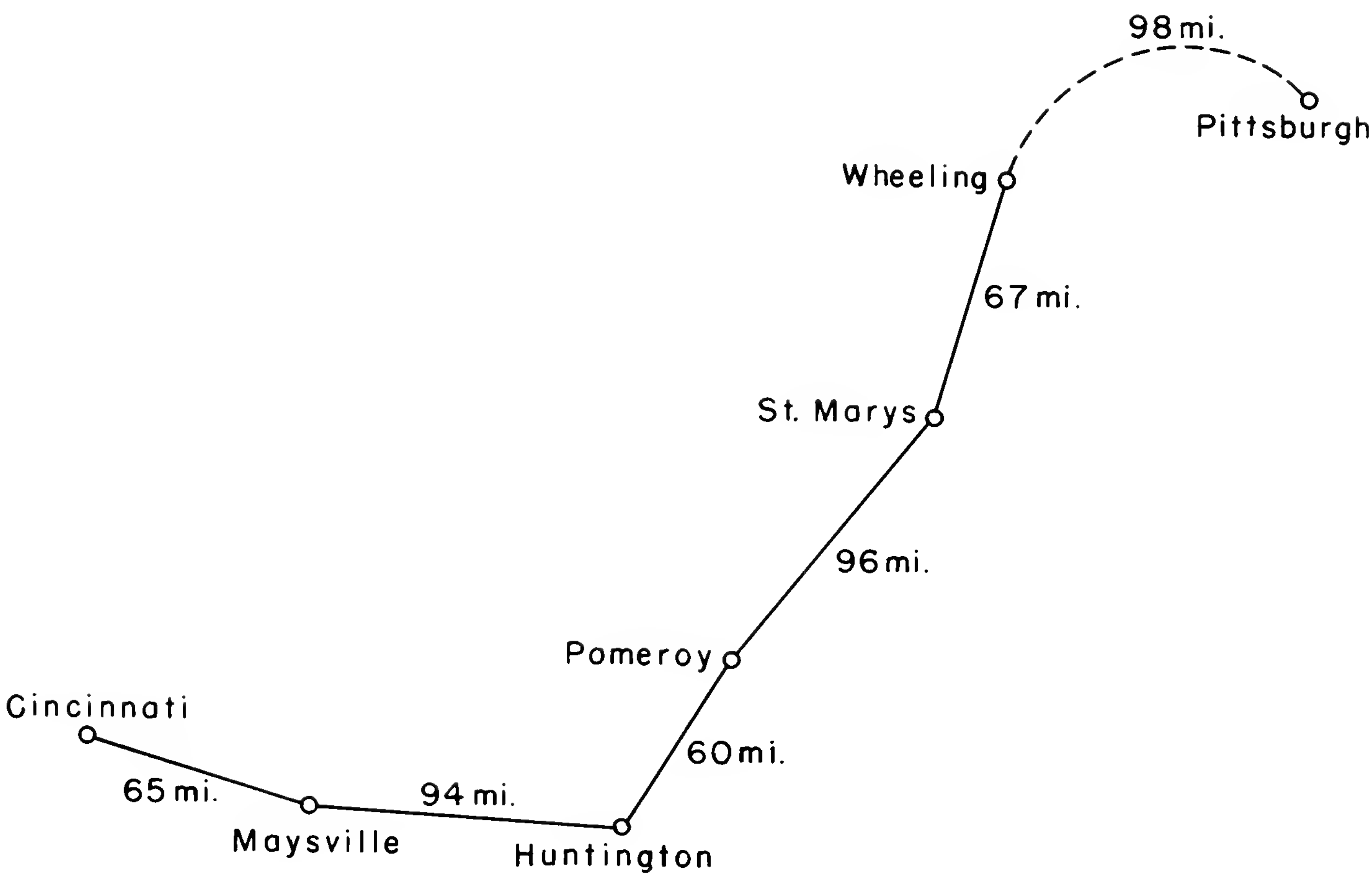


Fig. 3.1

However, from analyses of the records of past floods (which included discharge as well as stage measurements at the gaging stations mentioned above) the storage volumes in each reach were known as a function of the discharge at the lower end of the reach. (The data were available in this form because it is in this form that it is used in the conventional flood routing procedure.) In addition to the storage volume as a function of discharge at the lower end of the reach there is also available the rating curve, that is, the relation between discharge and stage at a gaging station for steady flow conditions in the river. (Since actual flows are rarely steady some manipulation of the observational material is necessary in order to obtain a rating curve for steady flow conditions). Next it is necessary to know the average slope of the water surface over a reach for steady conditions as a function of the stage* at the lower end of the reach —— which can be obtained from past flood records. It is therefore possible to calculate the stage at the middle of the reach as a function of the discharge at the lower end of the reach, whence the storage volume is known for the reach as a function of stage at the center of the reach. One need only divide the storage volume by the length of the reach to define the average crossection area as a function of stage at the midpoint of the reach. This was done for each of the five reaches in the stretch between Wheeling and Cincinnati. However, as indicated above, this gives averages at points seventy to eighty miles apart while a cross section area is needed for the finite difference scheme at points only 5 miles apart. It should be remembered that the mesh width is ten miles, but we use a staggered net. The areas at the net points were obtained by linear interpolation (as described below) between the midpoints of adjacent reaches.

In principle it would be possible to code the data for the cross section area as a function of stage and location as a two-entry table. In practice it is better, however to

*In Appendix II to Report II it is seen that the slope in a steady flow in a given channel is related in a unique way to the stage.

approximate cross section areas at the midpoints of the reaches by explicitly given functions involving a few parameters so that the digital computer can calculate the desired values from such simple formulas: in this way the storage capacity of the digital computer is conserved. The first method of approximating cross section areas tried by us was to use the following quadratic approximation formula

$$A(x,H) = a(x)[H - H_o(x)]^2 + A_o(x) \tag{3.1}$$

in which $a(x)$, $H_o(x)$, and $A_o(x)$ were first evaluated at the midpoints x_i of the reaches under study. This formula for the area at the midpoints of the reaches was then extended to the intermediate netpoints simply by interpolating linearly between the midpoints of two neighboring reaches. That is, the quantity $a(x)$ for example is assumed given as follows:

$$a(x) = \frac{x_1 - x}{x_1 - x_o}\, a(x_o) + \frac{x - x_o}{x_1 - x_o}\, a(x_1), \tag{3.2}$$

in which x_o and x_1 are midpoints of two neighboring reaches and x locates any intermediate net point. Similar formulas were used to fix the quantities $H_o(x)$ and $A_o(x)$ at intermediate points. For net points which were not located between midpoints of two reaches, as in the vicinity of boundary points, extrapolation formulas of the same sort were used with respect to the two nearest reaches. (In the case of the junction problem data from reaches outside the boundary points were available and thus in this case the boundary points played no special role. A slight modification in this process of fixing a cross section area was necessary at the junction, in order to make sure that the sum of the cross section areas of the Upper Mississippi and the Ohio equaled the cross section area in the Lower Mississippi.)

Once the cross section area has been fixed as a function of the stage, the width B is determined from the well known

relation $dA = BdH$ or by setting $B = \frac{dA}{dH}$. Figure 3.2 indicates the actual average cross section area for the reach from St. Marys to Pomeroy and also shows how the area is approximated by the quadratic formula (3.1)(a third curve given by a hyperbolic formula is also drawn——see discussion below). Figure 3.3 shows the corresponding width B determined by differentiation of the above area curves. As one sees - and this is quite important from the point of view of the later discussion - the empirical curve for the width B is approximated by the straight line obtained by differentiation of the parabolic area curve. In the upper Ohio this approximation to the width B is not accurate at the higher stages, especially in the reach Pomeroy-Huntington: the width increases at a much greater rate. The natural consequence was that the river stages obtained by calculation at Pomeroy were much too high, as is shown by Fig. 3.4, which shows observed stages there compared with those calculated using the parabolic approximation formula for areas.

These observations indicate the need for a more accurate approximation formula for cross section areas in at least some parts of the Upper Ohio. Since the linear variation in width as a function of stage furnished widths that appeared to be too small at higher stages, it was thought better to make use of a hyperbolic rather than a parabolic approximation formula for cross section areas. The formula finally fixed upon for A was the following:

$$A(x,H) = \frac{a(x)}{H-h(x)} + b(x)[H-h(x)] + A_o(x), \tag{3.3}$$

and hence $B = dA/dH$ is given by

$$B(x,H) = b(x) - \frac{a(x)}{[H-h(x)]^2} . \tag{3.4}$$

Upon comparison with (3.1) we note that there are now four parameters in (3.3), i.e. $a(x)$, $b(x)$, $h(x)$, and $A_o(x)$ instead

AVERAGE AREA FOR THE REACH FROM ST. MARYS TO POMEROY

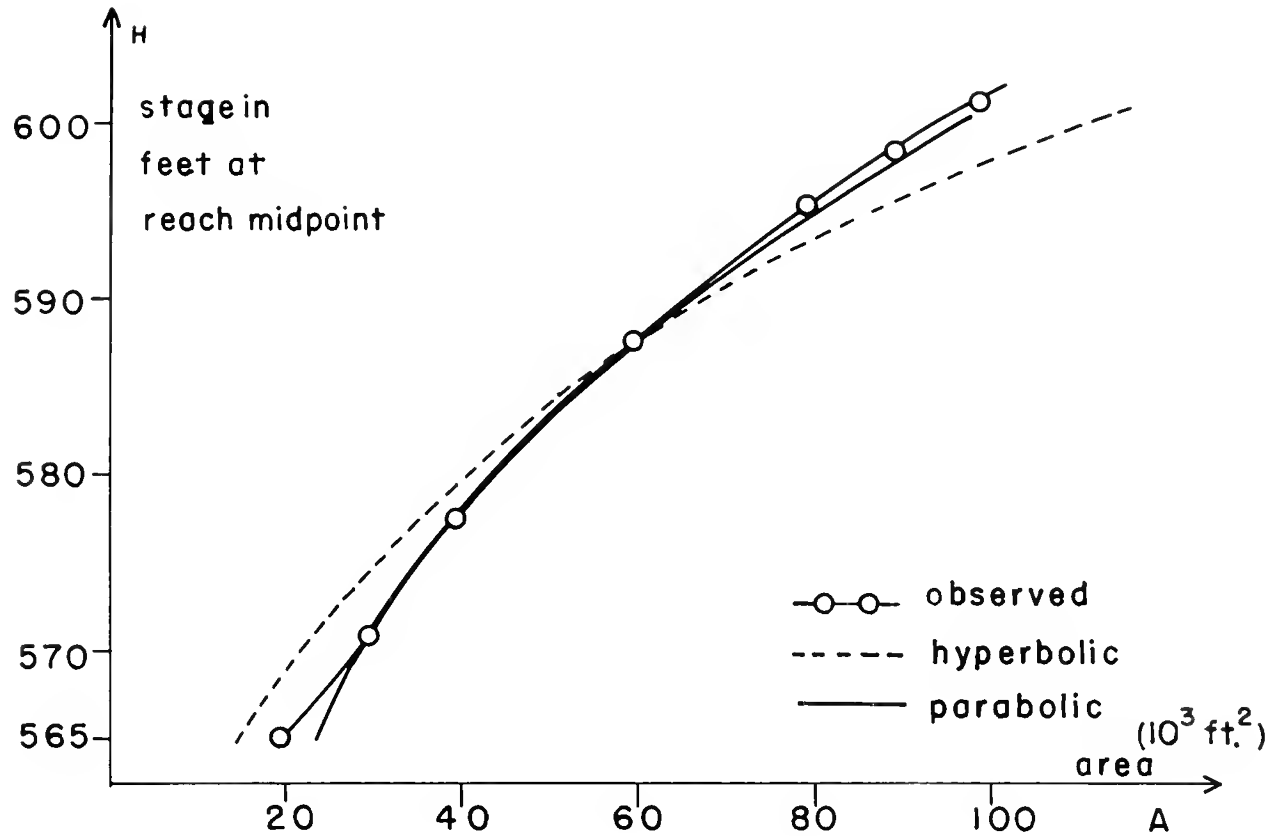


Fig. 3.2

AVERAGE WIDTH FOR THE REACH FROM ST. MARYS TO POMEROY

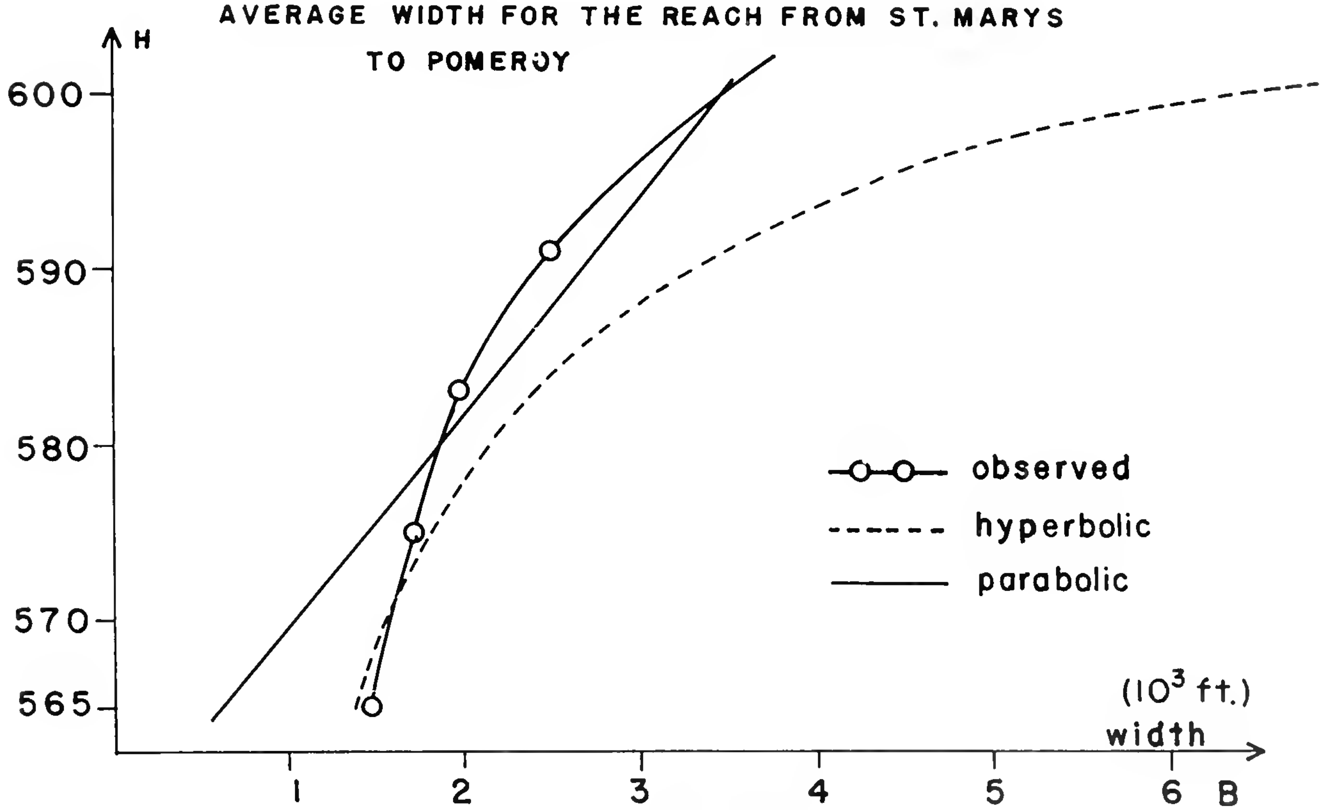
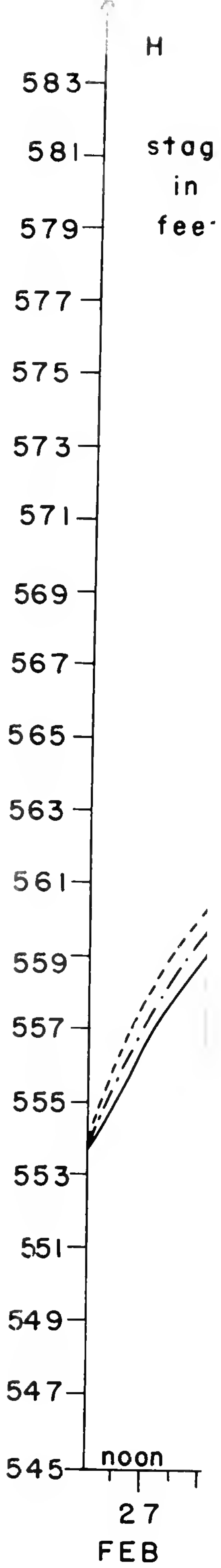


Fig. 3.3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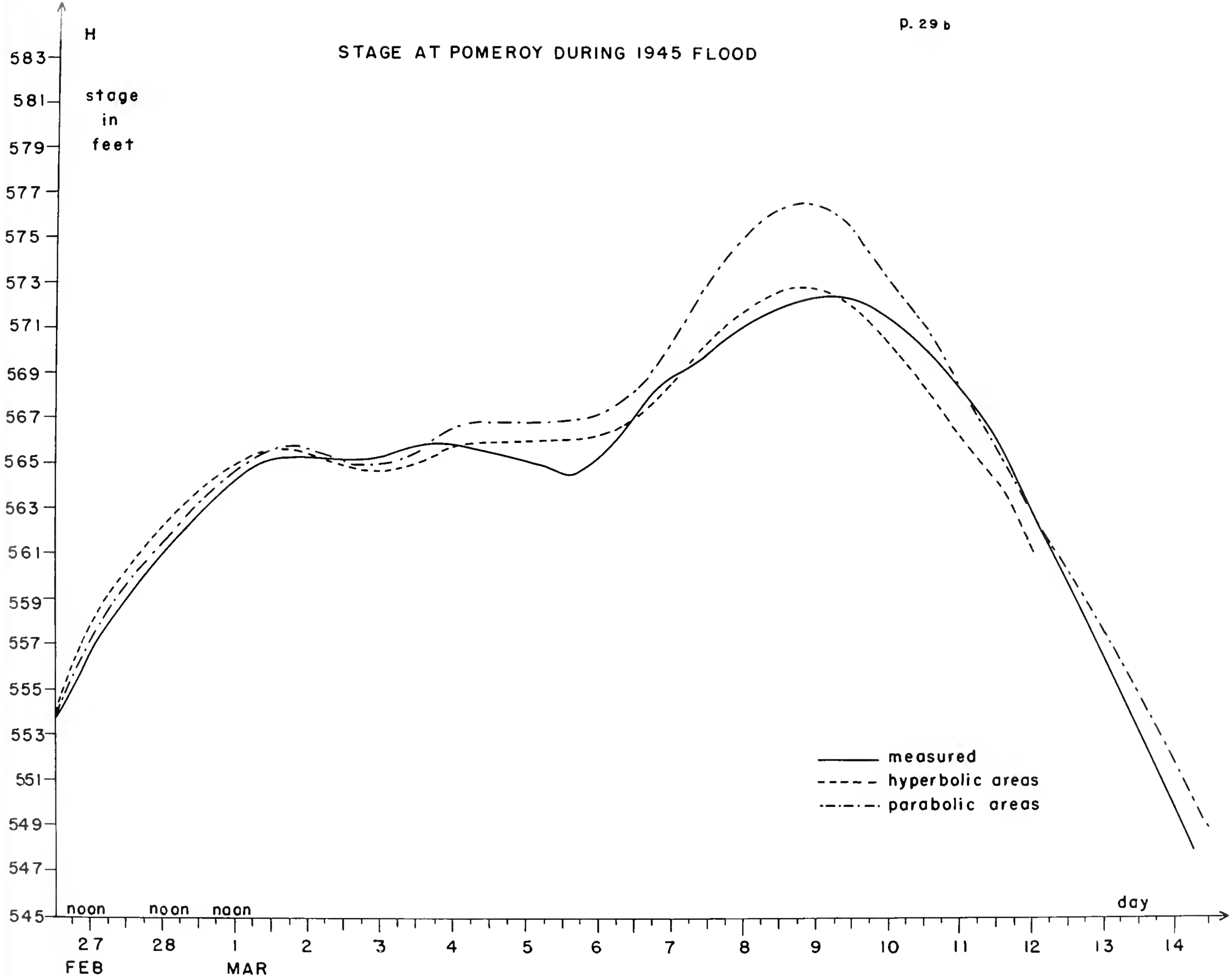
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hyperbolic areas
parabolic areas

Fig. 3.4

of the three parameters in (3.1). The use of these more accurate formulas (see Figs. 3.2 and 3.3) led to much better results, as is shown in Fig. 3.4, where the observed stages at Pomeroy are compared with those calculated using the two different formulas for cross section area.

For the problem of the junction of the Ohio and the Mississippi Rivers it was found sufficient to use a parabolic interpolation formula for the area---that is, formula (3.1) was used. Also, storage volumes for intervals of about 20 miles in length were given as part of the basic data, so that accurate average cross sections were known for intervals of this length.

In Kentucky Reservoir also, the average cross section areas were simply obtained from storage volumes over 10 mile intervals. These were obtained by planimeter from topographic surveys. However, quadratic approximation formulas for cross section areas were not accurate enough in the upper parts of the reservoir; in these portions the following cubic approximation formula was used, with good results:

(3.5) $A(x,H) = a(x)[H-h(x)]^3 + c(x)[H-h(x)] + A_o(x)$, with

$a(x)$, $h(x)$, $c(x)$ and $A_o(x)$ as parameters.

It should perhaps be emphasized that all of these approximation formulas for the cross section A refer, in the first instance, to a point at the center of a given reach (in the Ohio there were only five such reaches of varying lengths and in the other cases the reaches were approximately ten to twenty miles in length). Afterwards linear interpolations between midpoints of successive reaches were used to obtain the corresponding formulas at all netpoints in the manner exemplified by formula (3.2) for the parameter $a(x)$: that is, the linear interpolation was carried out <u>with respect to each coefficient separately</u>, and not to the formula as a whole.*

*In the upper Ohio, however, where the hyperbolic interpolation formula was used, it was necessary to modify this process somewhat because the reaches were very long and the parameters enter in a nonlinear way. In some cases additional intermediate cross sections were inserted before linear interpolation was carried out.

Finally, it might be added that our experience indicates that the approximation formulas should be checked carefully at the lowest stages, where they seem to be sensitive: otherwise, it can happen that a formula which yields reasonable approximations for the medium and high stages may yield absurd values (even negative areas) at very low stages.

b) Determination of the resistance coefficient $G(x,H)$:

We proceed to discuss methods of determining the resistance coefficient G which, of course, is also needed at all of the net points along the river as a function of the stage H. One way to determine this function would be to calculate it from hydraulic data, for example from the formula

$$G = \frac{gn^2}{(1.49)^2 R^{4/3}} \quad (3.6)$$

In this formula n is Manning's roughness coefficient and R is the hydraulic radius. A more direct approach, however, was preferred and equation (3.6) was used only as a check in order to see whether reasonable roughness coefficients resulted from the empirically determined values of G.

The second differential equation of (2.1) itself could be used as a means to define G if all other quantities in the equation were known. Fortunately, the terms v_t, vv_x, and $\frac{q}{a}v$ are in general negligible in value compared with the other two at any one instant of time* so that G can be simply computed from the formula

$$G = \frac{gH_x}{v^2}. \quad (3.7)$$

This formula rightly says that flows in large rivers are very nearly steady flows in which the velocity adjusts itself in such a way that the force of gravity down the sloping bed of the stream is very closely balanced by friction and turbulent resistance. From past records of floods the slope H_x is known, and the velocity v can be computed from the discharge records and the mean cross section area at the middle of the

*Their combined value is almost always less than 1% of the other terms.

reach in the manner described above. Formula (3.7) then furnishes an average value for G as a function of the stage H for each of the reaches. This was the method used to fix the coefficient G for the upper Ohio from records of past floods. As with the cross section area A, it is convenient to represent the resistance coefficient G by an approximation formula. For the Ohio River the following formula was used:

$$(3.8) \qquad G(x,H) = \alpha(x)[H-h_0(x)] + \frac{\beta(x)}{[H-h_0(x)]} + G_0(x),$$

where $\alpha(x)$, $\beta(x)$, $h_0(x)$ and $G_0(x)$ are first determined at the midpoints x_i of the reaches so that (3.8) approximates the average values obtained from (3.7). For other values of x, linear interpolation formulas of the type of (3.2) are applied to the coefficients $\alpha(x)$, $\beta(x)$, $h_0(x)$, and $G_0(x)$ as in the case of the area coefficients. In figure (3.5) we plot, as an example, the average resistance coefficient G for the reach from Pomeroy to Huntington. (The resistance curve is afterwards adjusted on the basis of trial calculations as explained in section 4).

For the case of the junction problem and also for Kentucky Reservoir the determination of G was simplified because the basic data furnished by the engineers included the knowledge of what is called the conveyance factor K, which is related to our coefficient G by the formula

$$(3.9) \qquad G = \frac{1}{K^2} \cdot \frac{ga^2}{L}$$

in which L is the length of the reach for which the conveyance factor K is known. For the case of the junction and Kentucky Reservoir problem, K was known for reaches approximately ten miles in length. The curves obtained from (3.9) were then approximated by a parabolic formula. At intermediate net points linear interpolation was again used to fix the values of G.

AVERAGE RESISTANCE COEFFICIENT G(H) FOR REACH FROM POMEROY TO HUNTINGTON

a – determined from 1945 flood

b – suggested by engineers c – adjusted

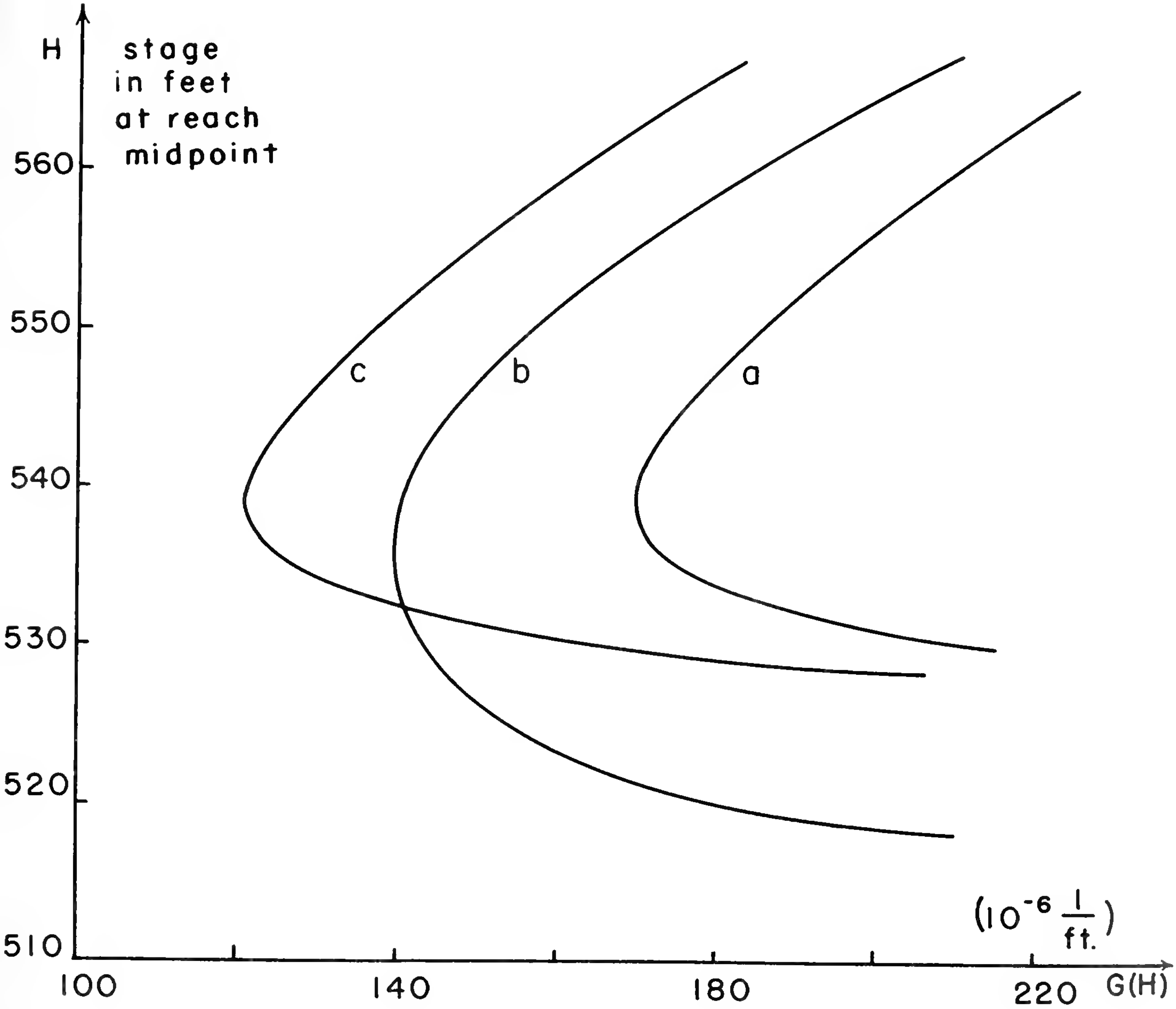

Fig. 3.5

The best check on the correctness of our coefficients is of course obtained by checking the calculated stages and discharges against the observed quantities in actual floods. However, in the case of the resistance coefficient G a check for general consistency is possible using formula (3.6). In this formula the hydraulic radius R is a purely geometrical quantity which is easily computed from the known cross section areas as functions of the stage (as described above): it differs, in fact, in all of our cases very little from the mean depth A/B. Hence the formula (3.6) makes it possible to calculate Manning's coefficient, the roughness factor n. This quantity should vary between .01 and .15, but for rivers of the sort we are interested in its value should be somewhere near .03. In all of our three cases reasonable values for n were obtained, as follows:

Ohio River $.02 \leq n \leq .05$ (mostly $.025 \leq n \leq .035$)
Junction $.015 \leq n \leq .035$
Kentucky Reservoir $.01 \leq n \leq .04$

Also, the roughness coefficient increased in general with stage, as it should.

The fact that Manning's coefficient does not vary a great deal is a fact that might be used to study flows in rivers for which the data needed to determine our coefficient G are insufficient: a first estimate of it could at least be obtained, which might then be improved gradually once more information on floods became available.

c) Preparation of initial and boundary data:

It has been stated a number of times that it is necessary to know the state of a river at some initial instant, taken by us to be at $t = o$, and this implies that the stage H and velocity v are known along the river initially. With the stage H there is no difficulty: it is directly given. The determination of the initial velocity v in general requires some calculation using the basic data.

In the case of the Ohio River and also in the case of the junction problem it was assumed that the initial state was near enough to a steady flow that the second equation of (2.1) could be used to determine a value for v at each net point, now that values for G have been determined by the methods described above. (Some experiments were made with the Univac by using the <u>observed</u> initial velocities, and it was found that results would be influenced only slightly since the initial errors were smoothed out within an hour.) This means that v is fixed initially by the calculation from the formula

$$(3.10) \qquad v^2 + \frac{gH_x}{G} + \frac{q}{A} v = o,$$

with q the local run-off and tributary inflow for the particular net point. Actually, the term qv/A is not important.

In Kentucky Reservoir discharge measurements are available at the ends of the reservoir only, i.e. at Kentucky Dam and Pickwick Dam. In this case the velocity at the two end points was of course fixed by the formula $v = Q/A$. A linear interpolation of the discharge between the two ends then furnished the initial values of the discharge at intermediate net points, and a division by the local cross section area then furnished the initial velocities at the intermediate net points.

In the upper Ohio, the boundary conditions assumed by us were that the stage was known as a function of time at Wheeling and Cincinnati. At Cincinnati, the down-stream end, it is of course somewhat artificial to prescribe the stage---it should rather be computed as one of the important unknowns as was done in the junction problem which we describe next. In the problem of the junction of the Ohio and Mississippi Rivers the boundary data were applied in two different ways, with good success in both cases. In the first calculation the stages at the upper ends of the Ohio and the Mississippi were prescribed, but in the second calculation

these were replaced by the discharges, which seems to be the more natural procedure from the practical point of view. In both cases, however, neither stage nor discharge was prescribed at Hickman at the lower end of the Mississippi. Instead, the condition which is natural from the practical point of view at the lower end point of a portion of a river which continues open below that point was used there which furnishes a relation between stage and discharge.* In our calculations, an average rating curve was used that was the result of observations of past floods. In Kentucky Reservoir, which is closed at both ends by dams, the natural conditions were used, i.e. that the discharges were assumed known as functions of the time: these are the physical quantities which are subject to direct control.

d) Determination of the maximum permissible time step Δt:

As was explained in Report II, the maximum time step Δt which can be used to advance the solution from time t to time $t + \Delta t$ is fixed by the inequality

$$\Delta t \leq \frac{\Delta x}{v+c} \tag{3.11}$$

in which Δx is the half mesh width (in our staggered scheme of net points), v is the velocity, and c is the propagation speed of wavelets given by the formula

$$c = \sqrt{\frac{gA}{B}} = \sqrt{gy_m}, \tag{3.12}$$

with y_m the mean depth of the river.

It turned out that c varies between the same limits in all three of our cases, i.e. between 20 ft./sec., and 30 ft./sec. corresponding to average depths between 10 ft. and 40 ft., while the maximum value of v is of the order of 5 or 6 ft./sec. (It is thus to be noted that we are operating always with flows with velocities far under the critical velocity). Since Δx in (3.11) is five miles in our cases, it

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*It would perhaps be better to take rating curves with the slope H_x figuring as a parameter.

is readily found that Δt should be taken not greater than about 10 minutes. We have taken it to be 9 minutes to be on the safe side. This is of course a very short time step in comparison with the times, of the order of weeks, for which the flows have been computed. Nevertheless, it is not possible (at least not without a radical revision in the whole method of computation) to relax this condition---which, it will be remembered, is imposed by basic mathematical facts concerning our differential equations. In fact, if Δt is taken even slightly larger than the limit imposed by (3.11), the result is likely to be, not simply inaccurate results, but rather values of the unknowns which oscillate wildly. This experience--- well known in similar problems in other fields, particularly in gas dynamics---was verified for our problems by an empirical test using an exact solution of the differential equations as a basis for comparison. A steady progressing wave in a uniform channel (in fact, the wave described in Report II) was taken and initial values were chosen to conform to it; these were then advanced in the time by numerical calculation using our methods. When Δt was chosen properly, the results checked the known solution very well; however, when Δt was taken larger than the permitted value for convergence, wild oscillations occurred at once in the vicinity of those places where condition (3.11) was violated. The same experience was noted also in a simple model of the Kentucky Reservoir.

e) Further remarks about the preparation of data:

In this section we have described how the basic data for a river, river system, or a large reservoir can be used to furnish the coefficients and other data needed to formulate the flow problems in terms of differential equations. It should be remarked that the basic data for the most part are worked out for steady conditions, and that such data may not be sufficient in all respects for the purposes in view. The acid test is to check the results for actual floods against those

obtained by calculation. If the two do not check, this implies that there are errors in cross section areas, or resistance coefficient, or inflows, etc. These quantities should then be changed in such a way as to give better results. It turned out in the three problems treated by us that only minor revisions were necessary in two of the cases, but that extensive revisions in a third (the upper Ohio River) were needed. Actually, the method of numerical calculation has in it the inherent possibility of improving the basic data for a river by constant checking against the results for new floods, and making changes in coefficients where changes are indicated. In the next sections, where our final results for the three problems are presented, some details on this matter will be given.

§4. Comparison of Predicted and Observed Stages in the Upper Ohio River in the 1945 and 1948 Floods

In Fig. 3.1 of the preceding section a diagrammatic sketch of the Ohio River between Wheeling and Cincinnati is shown together with the reaches and gaging stations. By the methods discussed in the preceding section resistance coefficients and cross section areas representing averages over each reach are available. These in turn furnish by linear interpolation values at the net points of our finite difference scheme. An interval between net points of 10 miles in the staggered scheme described in §2 was taken, on the basis of the experience with a simplified model of the Ohio River which was presented in Report II. Calculations for the actual Ohio River for a limited period of time were made with a 5-mile interval, in order to get some idea of the possible errors to be expected from use of 10-mile intervals. The results at Pomeroy for a 36-hour period are shown in Fig. 4.1. There is a difference of 6 inches between the two, as we see - which is not entirely negligible. However, it was nevertheless decided to proceed on the basis of 10-mile intervals in order not to use too much calculating machine time.* A time interval of 9 minutes was used which is well under the maximum permissible for convergence of the finite difference scheme.

Flood calculations for the 1945 flood were begun at a time when the river was low. Calculations were first made for a 36-hour period during which the flood was rising; as stated earlier these calculations were made using the measured inflows of tributaries and the estimated runoff in the main valley. Upon comparison with the actual records it was found that the

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*It should be repeated here that we began the investigation which forms the subject of this report with the problem of the Ohio River, and had therefore to proceed without the aid of previous experience from any source. Realizing that it would doubtlessly be necessary to experiment with and to revise the methods of calculation, it was decided that to use too fine a net at the beginning would result in waste of expensive calculating machine time.

Calculated Stages for Pomeroy

a - using 5 mile intervals

b - using 10 mile intervals

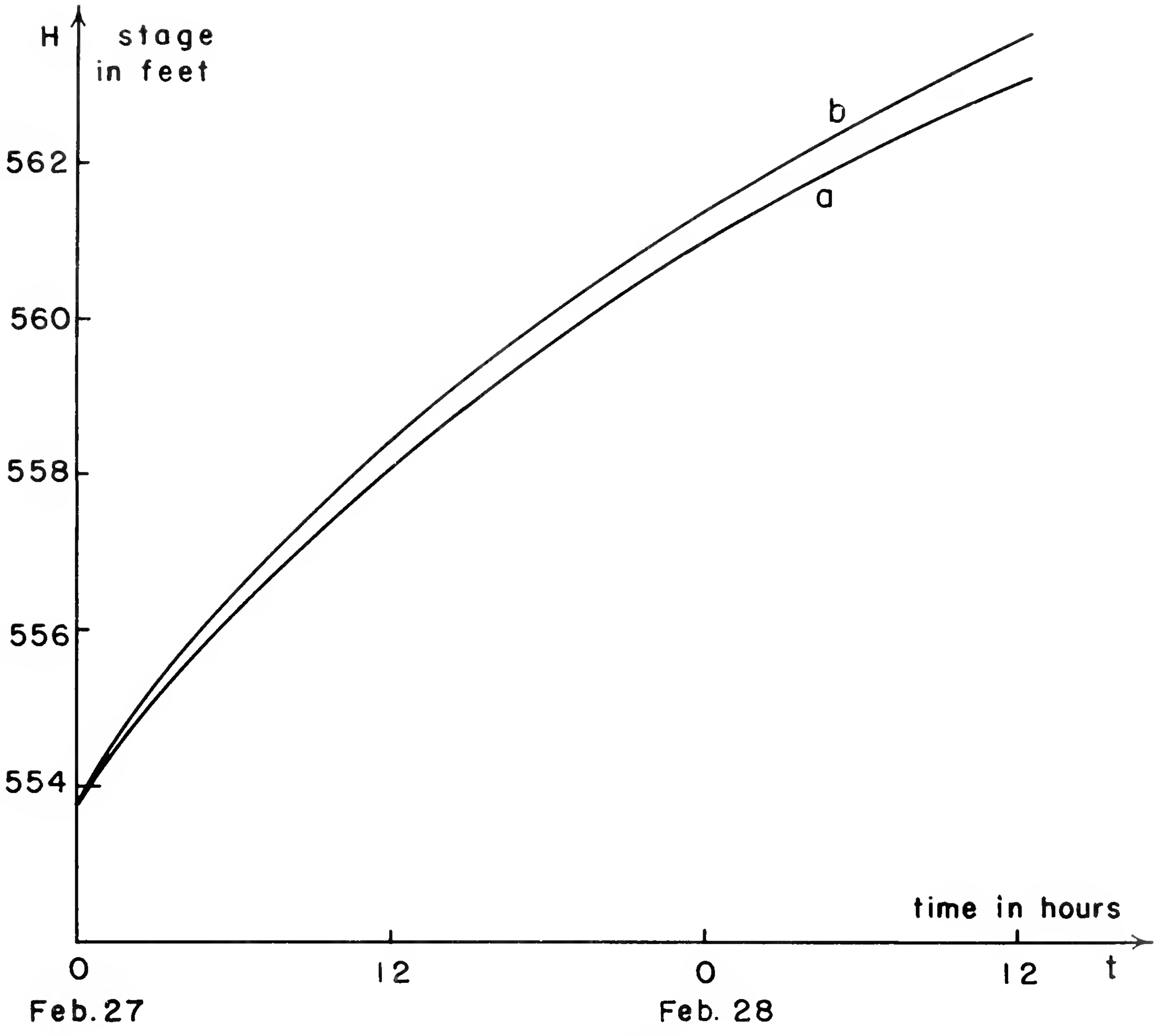


Figure 4.1

calculated flood stages were in general systematically higher than the observed stages and that the discrepancies increased steadily with increase in the time.

For example, Fig. 4.2a shows the calculated versus the observed stages for the first 36 hours at Pomeroy. It was reasonable to suppose that the deviation was probably due to an inaccuracy in the resistance coefficient. Consequently a series of flow calculations was made on the UNIVAC in which this coefficient was varied; from these results an adjusted resistance coefficient was estimated for each of the reaches. In Fig. 4.2b the results of two such computations are shown. In both cases, the resistance coefficient G was lowered substantially (a maximum of 20 % -30 %) over a portion of the river, followed by a recalculation of the flow for a 24-hour period. The curves show the difference in stages obtained when the resistance coefficient is changed. In case (i), the coefficient G was decreased considerably at and near Pomeroy, and the result was a change of stage of about 0.3 foot. In case (ii), the change in G was greater, and it extended over a larger portion of the river; the result was a much greater change in stage - a maximum of 1 1/2 feet - as was to be expected. In both cases one observes that the maximum lowering of the stage occurs somewhat upstream from the region where G is lowered, and is followed downstream with a somewhat smaller increase in stage. Lowering the resistance in one section seems to increase the flow above that section and to pile up the water downstream. Once the effects of changes in G have been estimated, it becomes possible to make the changes in such a manner as to bring calculated stages into agreement with observed stages. Actually this was done rather roughly, simply by shifting bodily the original curves for the coefficient G as a function of stage by a displacement of an appropriate amount in each reach (i.e., only the constant $G_o(x)$ in (3.8) was changed). In other words no attempt was made to make corrections that would require modificatio in the <u>shape</u> of these curves in their dependence on the stage.

Stages at Pomeroy

a - observed

b - calculated with original resistance coefficient

c - calculated with adjusted resistance coefficient

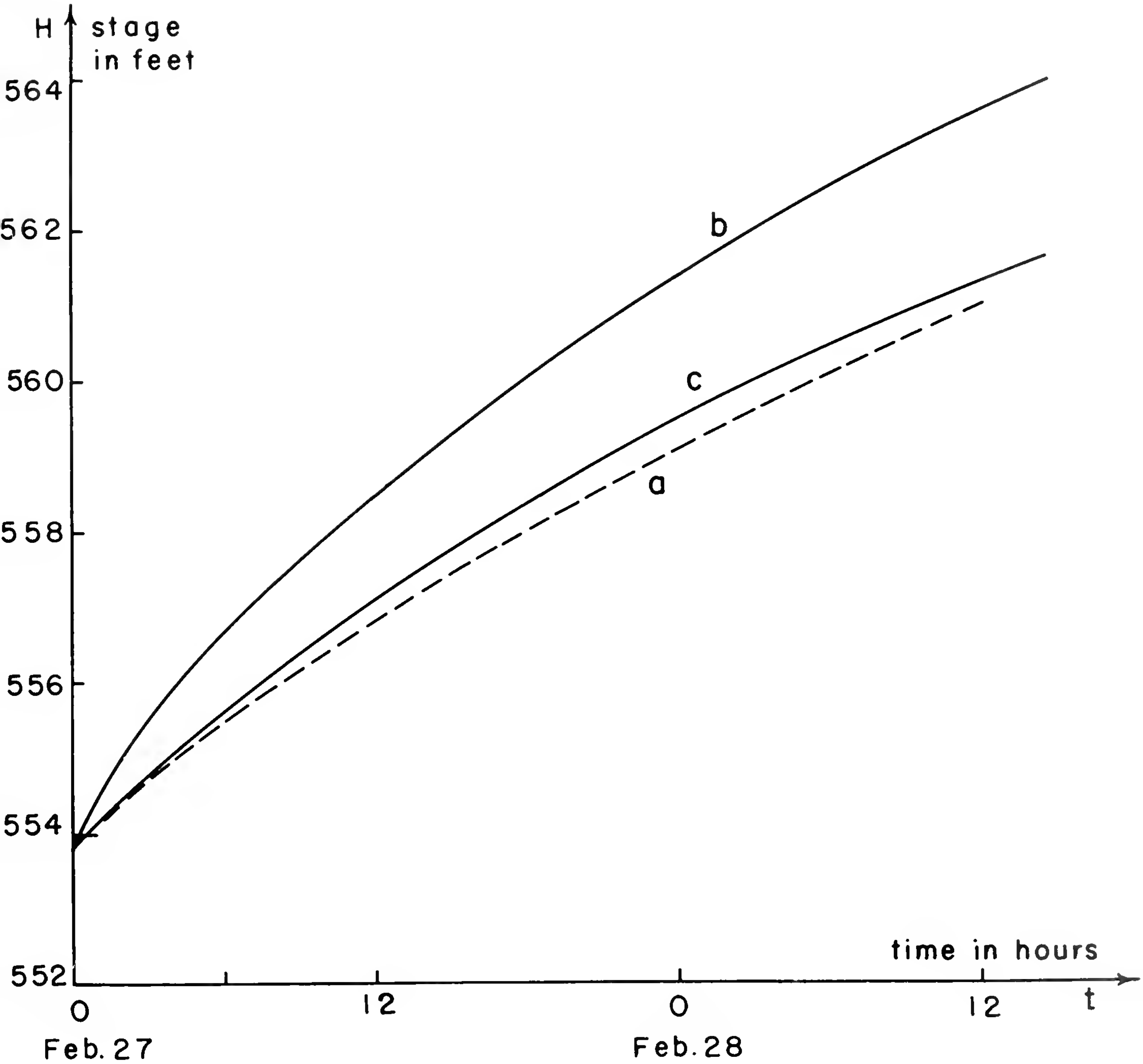


Fig. 4.2a

THE CHANGE IN STAGE, AFTER 24 HOURS, WHICH RESULTS FROM A CHANGE IN RESISTANCE COEFFICIENT

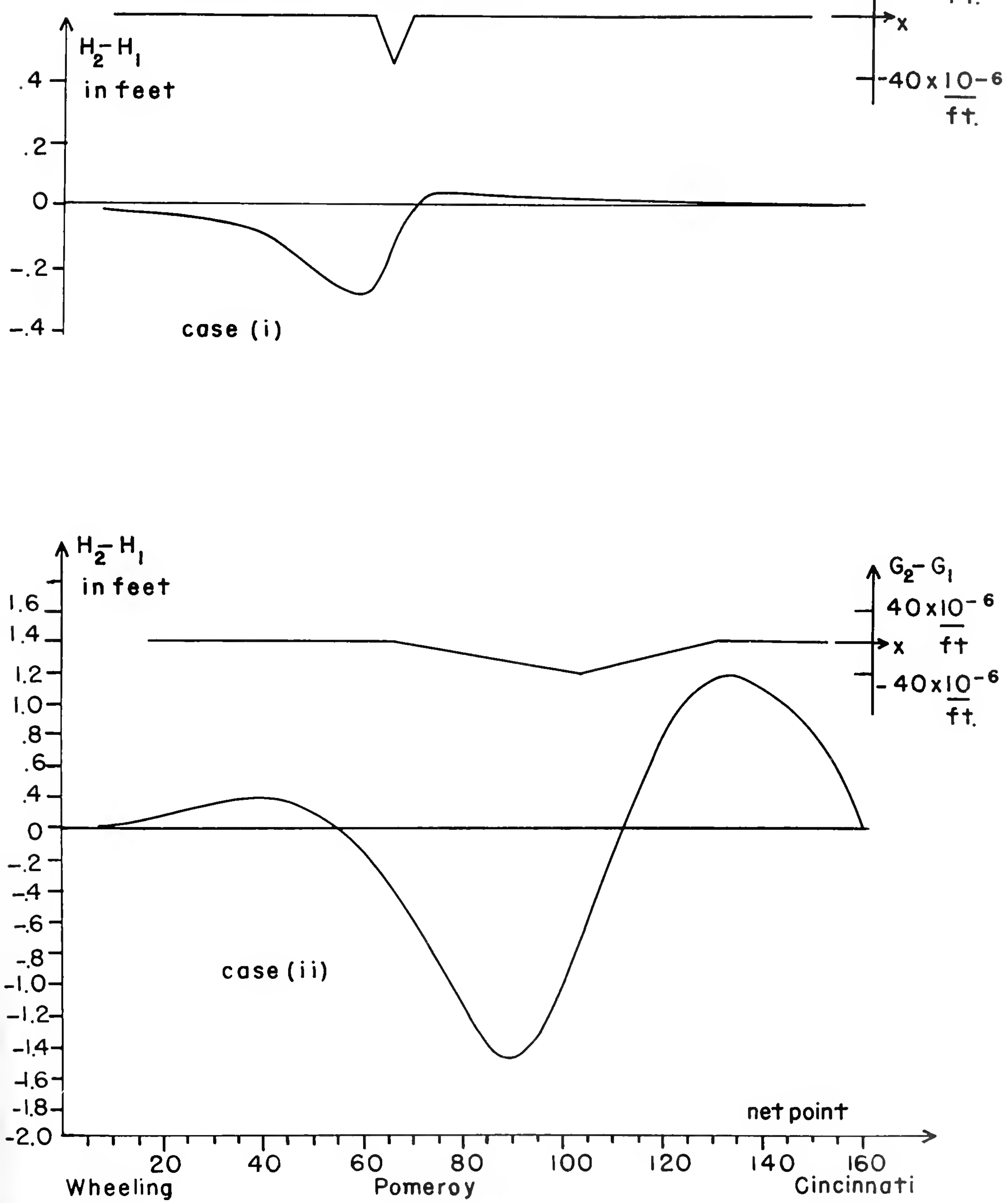


Fig. 4.2b

In Fig. 4.3 we plot the average resistance for the reach from Pomeroy to Huntington. We show the curve determined by the engineers from the basic data, the curve we found using formula (3.7), and the shifted curve determined by the trial calculations. In Fig. 4.2a we have already given the stages calculated with the original and the shifted resistance coefficients at Pomeroy for the first 36 hours. The new coefficients thus corrected on the basis of 36 hour predictions (and thus for flood stages far under the maximum) were then used to continue calculation of the flow for various 6 day periods as well as one 16 day period, with results to be discussed in a moment.

It should be said at this point that making such a correction of the resistance coefficient on the basis of comparison of results from a calculation of an actual flood with the observed quantities corresponds exactly to what is done in making model studies. Indeed, in making model studies no first estimate for the resistance is possible a priori as is the case with the method being described here; instead it is always necessary to make a number of verification runs after the model is built in order to compare the flood stages given by the model with actual floods. In doing so the first run is normally made without making any effort to have the resistance correct. In fact, the roughness of the concrete of the model furnishes the only resistance at the start. Of course, it is then observed that the flood stages are too low compared with an actual flood because the water runs off too fast. Brass knobs are then screwed into the bed of the model and wire screen is placed at other parts of the model to roughen it until it is found that the flood stages given by the model agree with the observations. This is in effect what was done in making numerical calculations except that the empirical data furnished at least a first estimate for the friction resistance in the river channel (for the junction and Kentucky

AVERAGE RESISTANCE COEFFICIENT G(H) FOR REACH FROM POMEROY TO HUNTINGTON
a–determined from 1945 flood
b–suggested by engineers c–adjusted

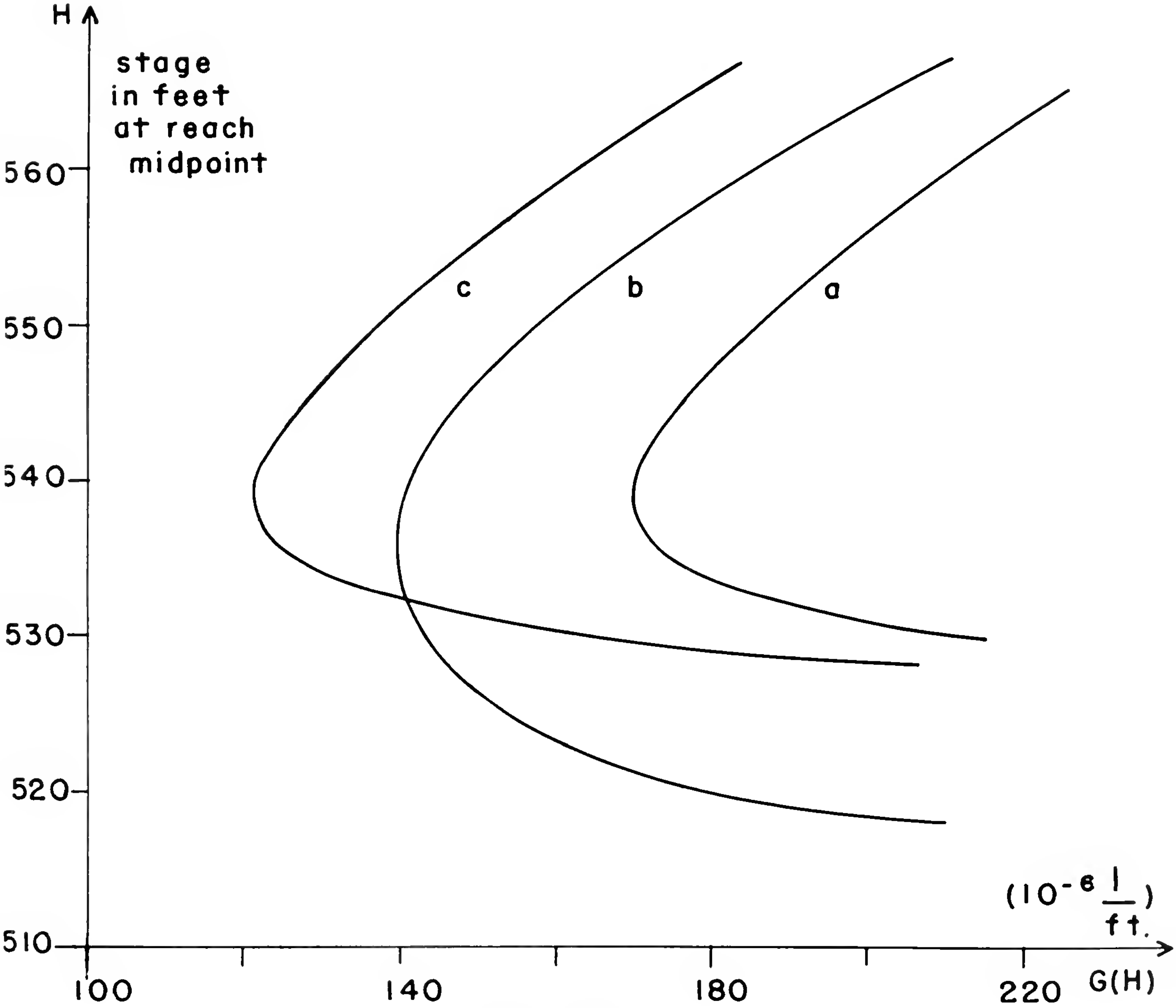
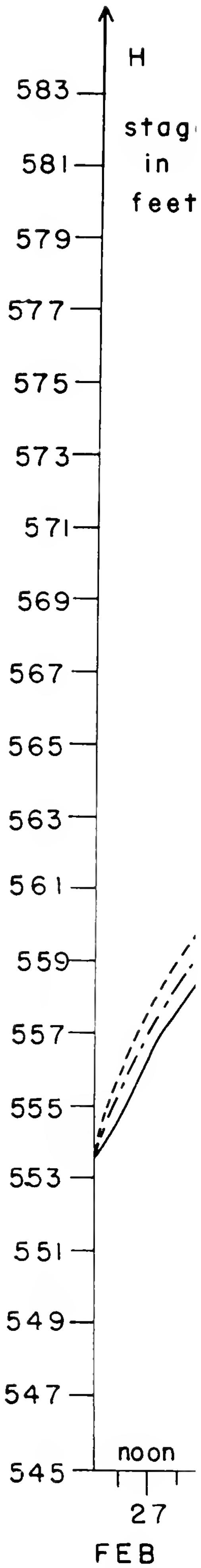


Fig. 4.3

Reservoir problems, the data on resistance and area were given for shorter reaches, and little adjustment of the resistance coefficients was necessary - see §§5 and 6).

In Fig. 4.4 the results of the first calculation of the 1945 flood are indicated. Shown in the figure are the river stages at Pomeroy. As was mentioned above, the resistance coefficients used were those obtained upon correction after making 36 hour trial runs. The cross section areas used were those obtained by the parabolic interpolation formula discussed in the preceding section; and as was already stated there, they proved to be not sufficiently accurate at the highest stages. One sees that the predicted flood stages approximate the observed stages with errors at the higher stages of somewhat less than a foot for the first 6 day period. At the crest of the flood, however, (on about March 9th, that is) after 11 days the error at the crest of the flood is more than 4 feet. As the flood recedes and stages become less the error once more becomes relatively small. That the calculated flood stages come out too high at the higher stages was judged to be due, as was explained above, to the fact that the actual cross section areas were apparently not well approximated at high stages by the quadratic formulas used by us. We therefore replaced the quadratic formulas by hyperbolic formulas and obtained much better results as can be seen in Fig. 4.4.

Upon using the hyperbolic interpolation formulas for cross section areas (again see the preceding section for details), the results for the 1945 flood were very much improved at all of the stations. In general, the errors in stage at the crest of the flood are small, of the order of a foot or less, and the maximum errors are of the order of 2 feet or less, as one sees from Figs. 4.5a, b, c, d which show the river stages at St. Marys, Pomeroy, Huntington, and Maysville. In addition, the minor variations in the observed stages are reproduced rather well by the calculated stages, though the latter in general tend to be smoother curves. This smoothing


H
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in
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581
579
577
575
573
571
569
567
565
563
561
559
557
555
553
551
549
547
545
noon
27
FEB

STAGE AT POMEROY DURING 1945 FLOOD

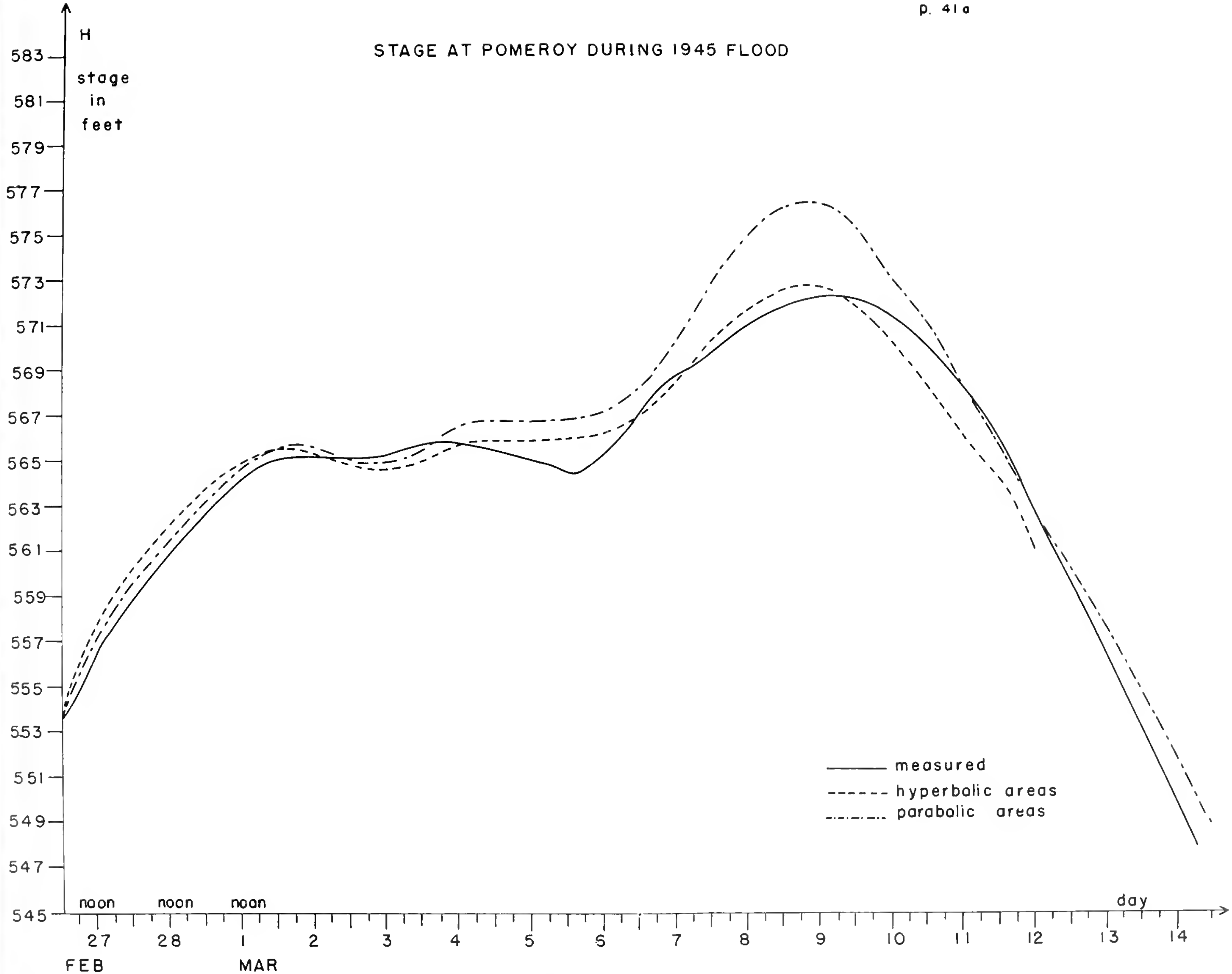
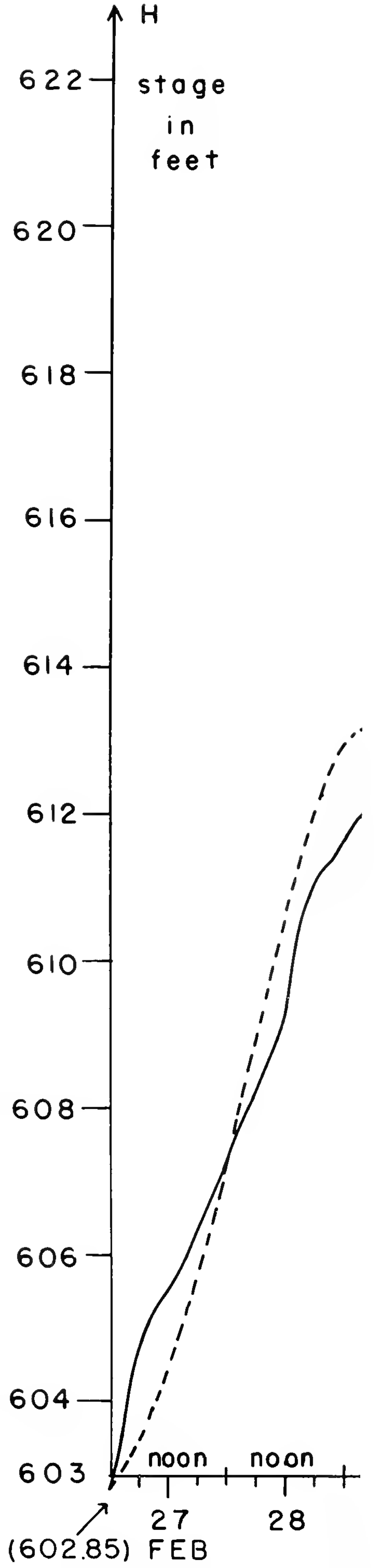


Fig. 4.4



H
stage
in
feet
622
620
618
616
614
612
610
608
606
604
603
noon
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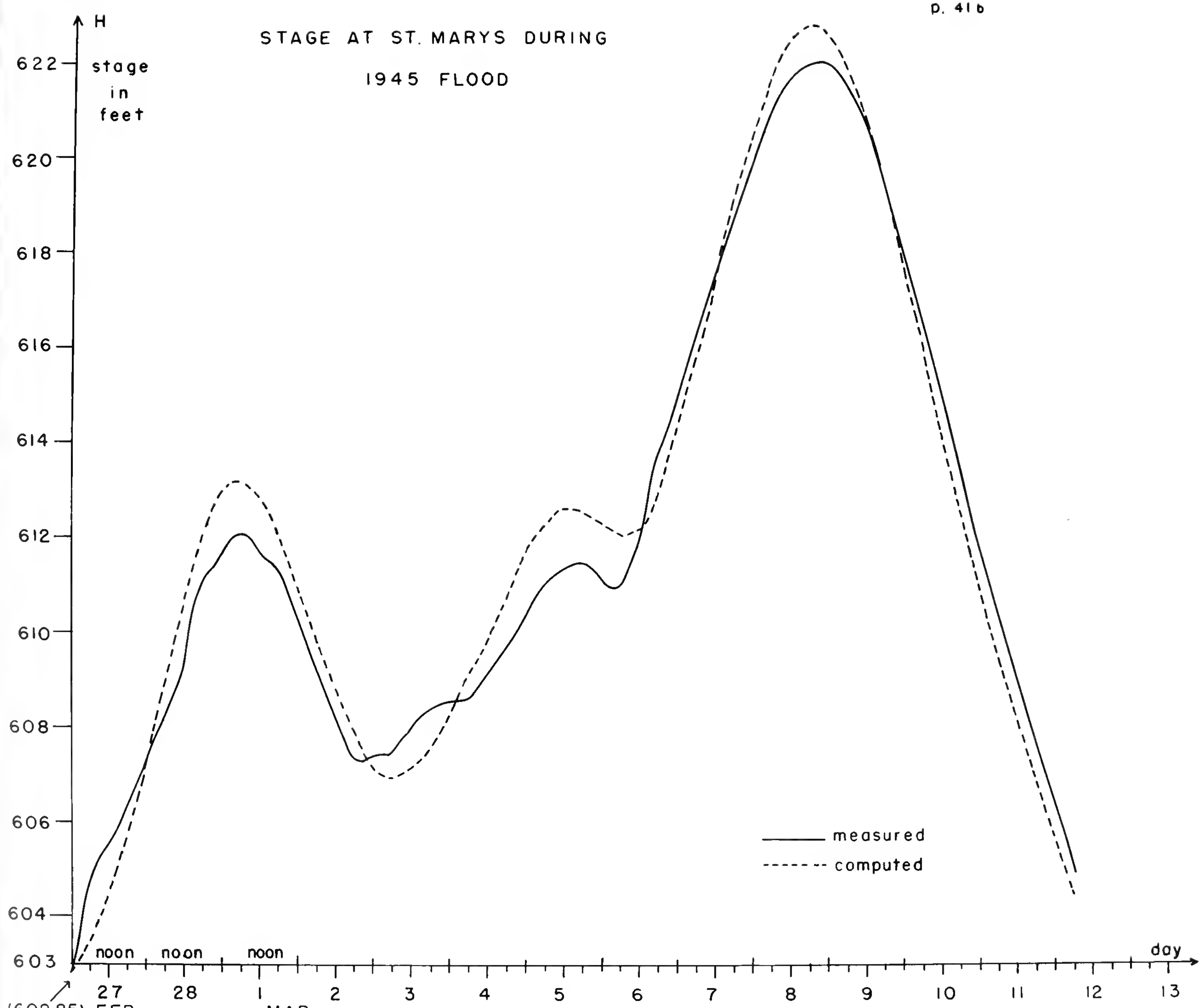
27
28
(602.85)
FEB


STAGE AT ST. MARYS DURING
1945 FLOOD
H
stage
in
feet
622
620
618
616
614
612
610
608
606
604
603
(602.85)
noon
noon
noon
day
27
28
FEB
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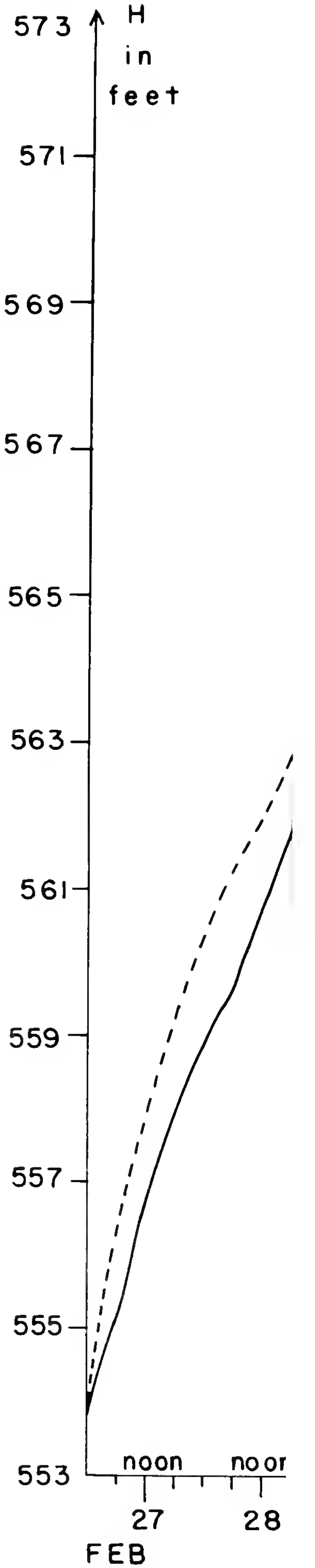
MAR
2
3
4
5
6
7
8
9
10
11
12
13
measured
computed

Fig. 4.5a



H in feet
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571
569
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565
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555
553
noon
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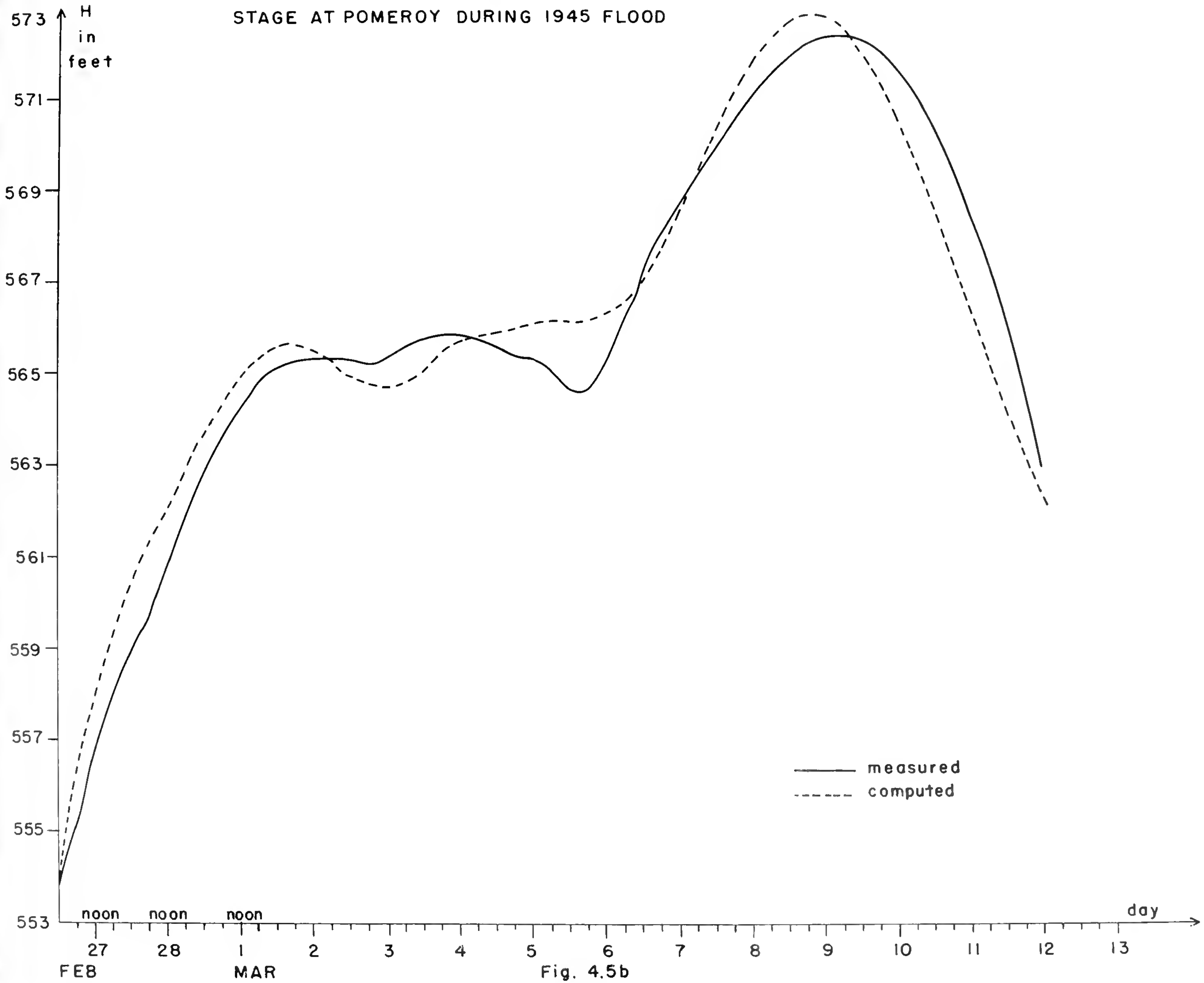
27
28
FEB


STAGE AT POMEROY DURING 1945 FLOOD
H in feet
573
571
569
567
565
563
561
559
557
555
553
measured
computed
noon
noon
noon
day
27
28
1
2
3
4
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6
7
8
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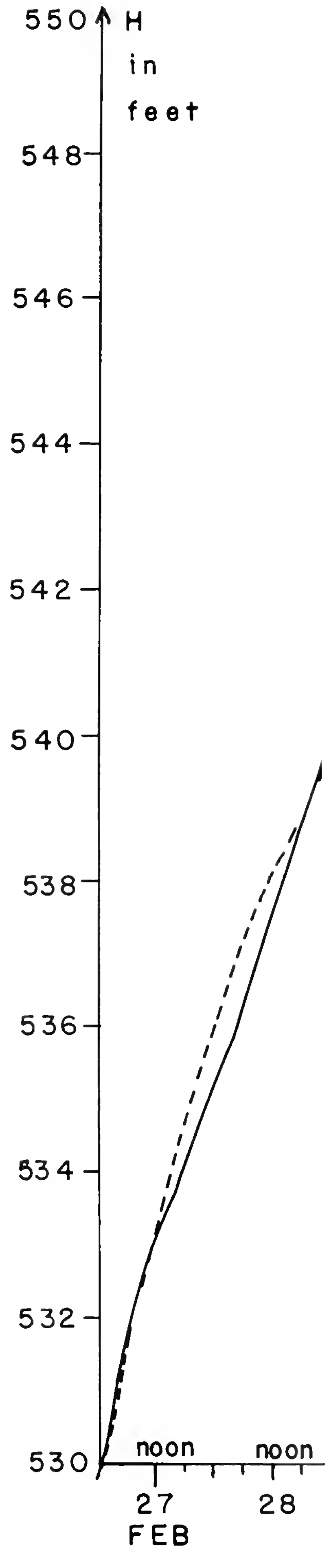
11
12
13
FEB
MAR

Fig. 4.5b



H
in
feet
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546
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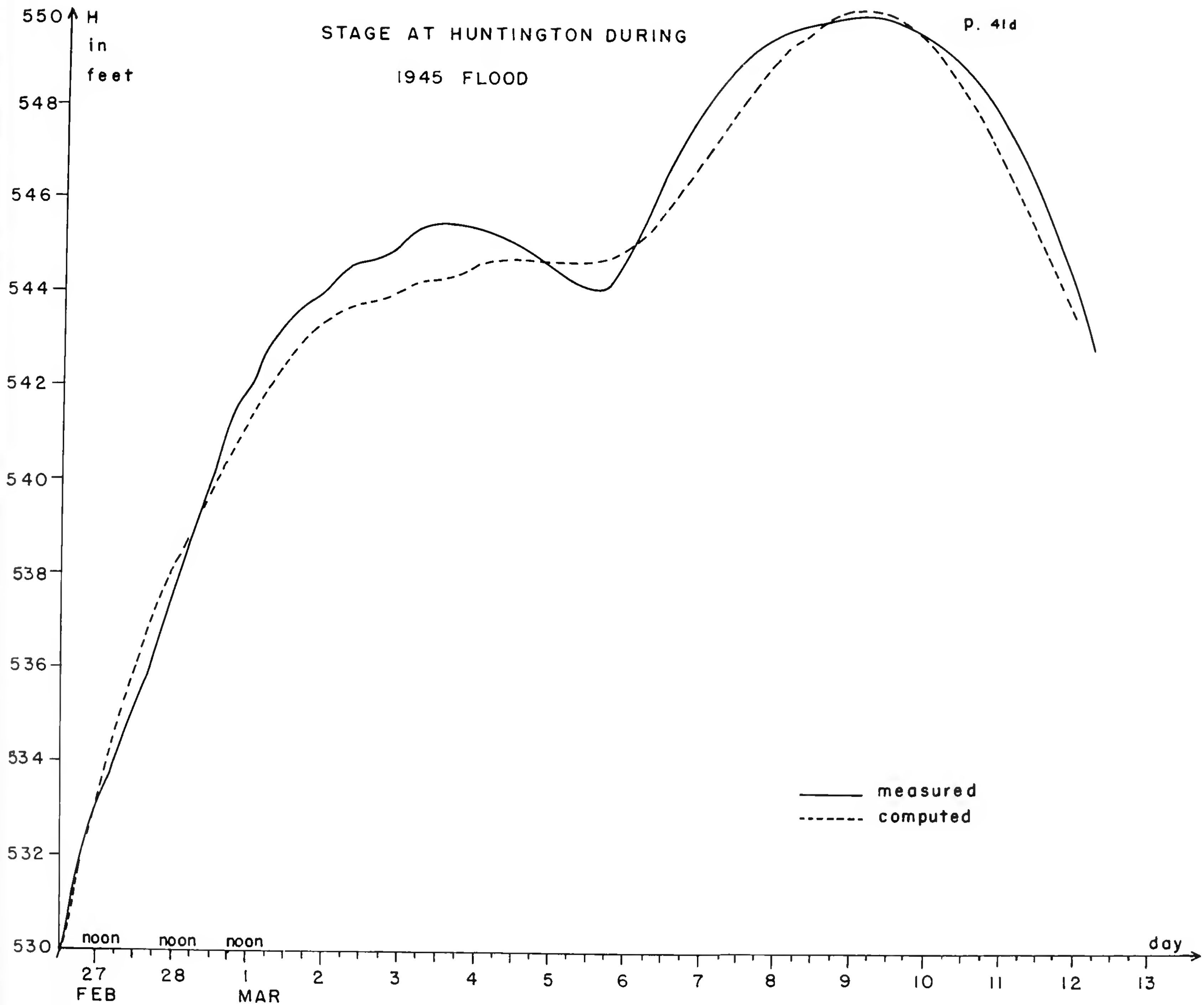
542
540
538
536
534
532
530
noon
noon
27
28
FEB


STAGE AT HUNTINGTON DURING
1945 FLOOD
H
in
feet
550
548
546
544
542
540
538
536
534
532
530
noon
noon
noon
day
27
FEB
28
1
MAR
2
3
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5
6
7
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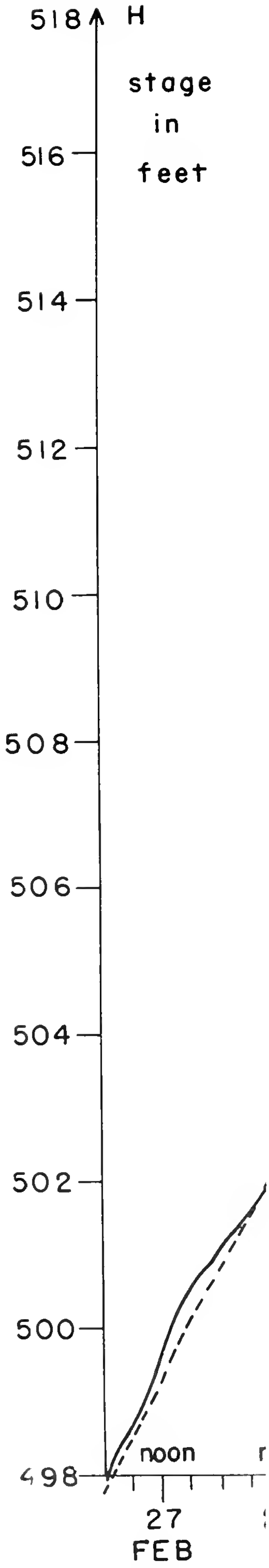
9
10
11
12
13
measured
computed

Fig. 4.5c


H
stage
in
feet
518
516
514
512
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508
506
504
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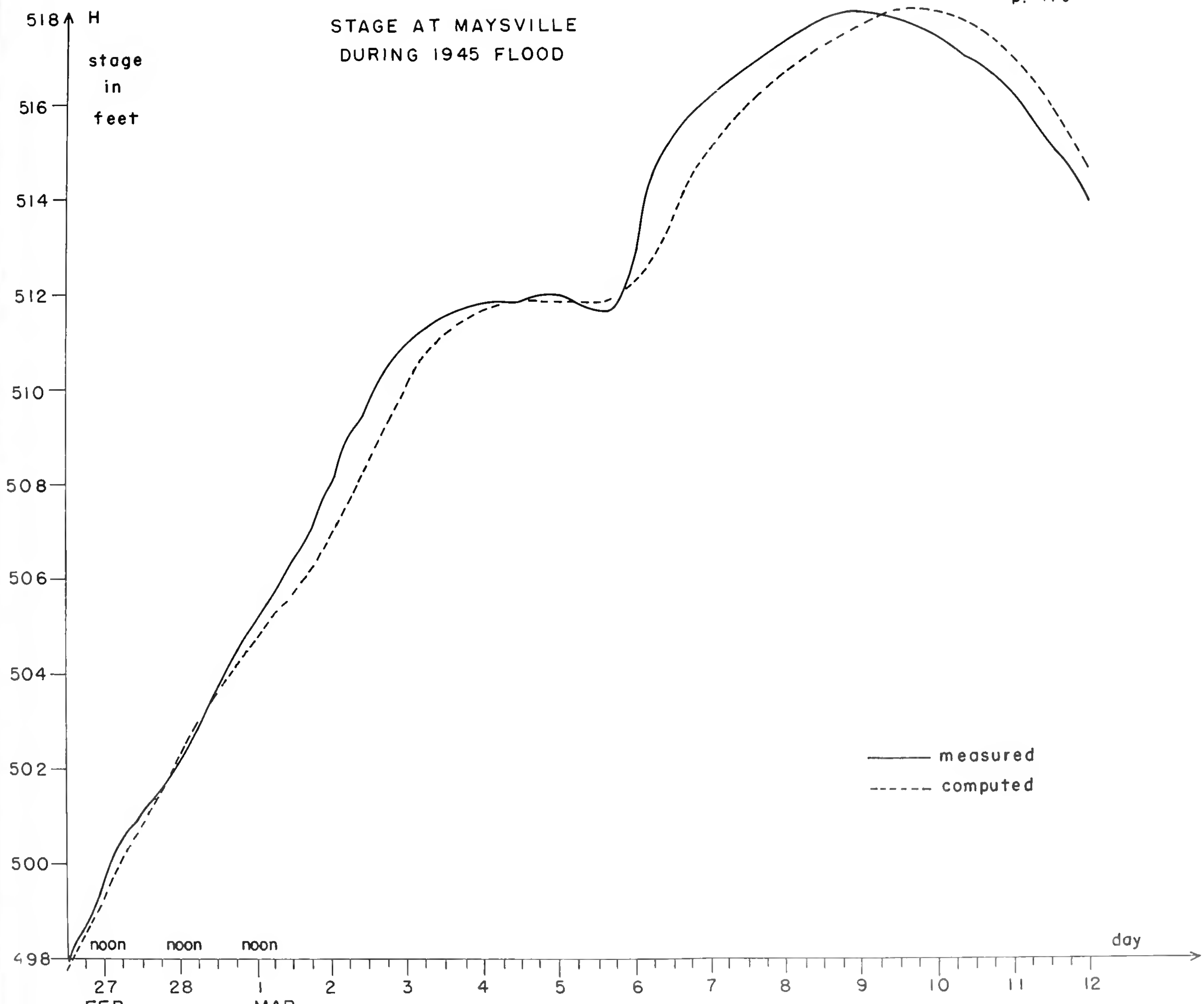
500
498
noon
27
FEB


STAGE AT MAYSVILLE
DURING 1945 FLOOD
518
H
stage
in
feet
516
514
512
510
508
506
504
502
500
498
noon
noon
noon
day
27
28
1
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3
4
5
6
7
8
9
10
11
12
FEB
MAR
measured
computed

Fig. 4.5d

effect, is, in some instances at least due to the distribution of the local inflow, which was done more smoothly in the calculation than is true in actuality (see the discussion in the introduction with reference to the hydrograph at Maysville, particularly near March 6).

By and large, it is right to say that the calculations reproduced the observed stages of the 1945 flood with quite reasonable accuracy after a good deal of adjustment of the cross section areas and resistance coefficients had been made. The necessity for such revisions of the coefficients obtained from the original data is a strong indication that the original data were not accurate enough - especially near Huntington. That this is true is then borne out by the results for the 1948 flood in the Ohio. If the coefficients used for the 1945 flood were correct it should be possible to calculate stages correctly for any other flood. However, upon doing so for the 1948 flood the results are not as accurate as they should be. Figure 4.6a, b, c, d shows calculated versus observed stages for the 1948 flood. All of these show the calculations for a 6 day period starting April 15. As one sees, the errors are high, particularly at Huntington, where the crest is wrong by about 1 1/2 feet. At other stations the results are better. Figure 4.7 shows the stages at St. Marys for a longer period starting April 12*: here the results are quite good. Much more work would be required to obtain more accurate coefficients for the Ohio, but there is no doubt that they could be obtained. In addition, various revisions in computational methods should be made, on the basis of experiences with the computations for Kentucky Reservoir. Further discussion of these points is postponed to §7, after the results for the junction problem and Kentucky Reservoir have been presented, and comparisons

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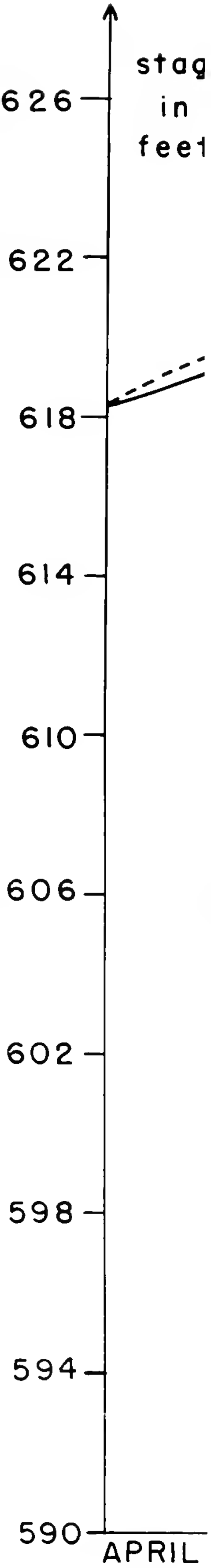
* The reason for the shorter period at the other stations is that the river was not open throughout the period in question - instead, a dam at Gallipolis, about 120 miles below St. Marys, was in operation, and this has considerable effects. Since St. Marys was rather far upstream it was thought reasonable to ignore the effect of the dam on the stages there.


stag
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feet
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622
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606
602
598
594
590
APRIL

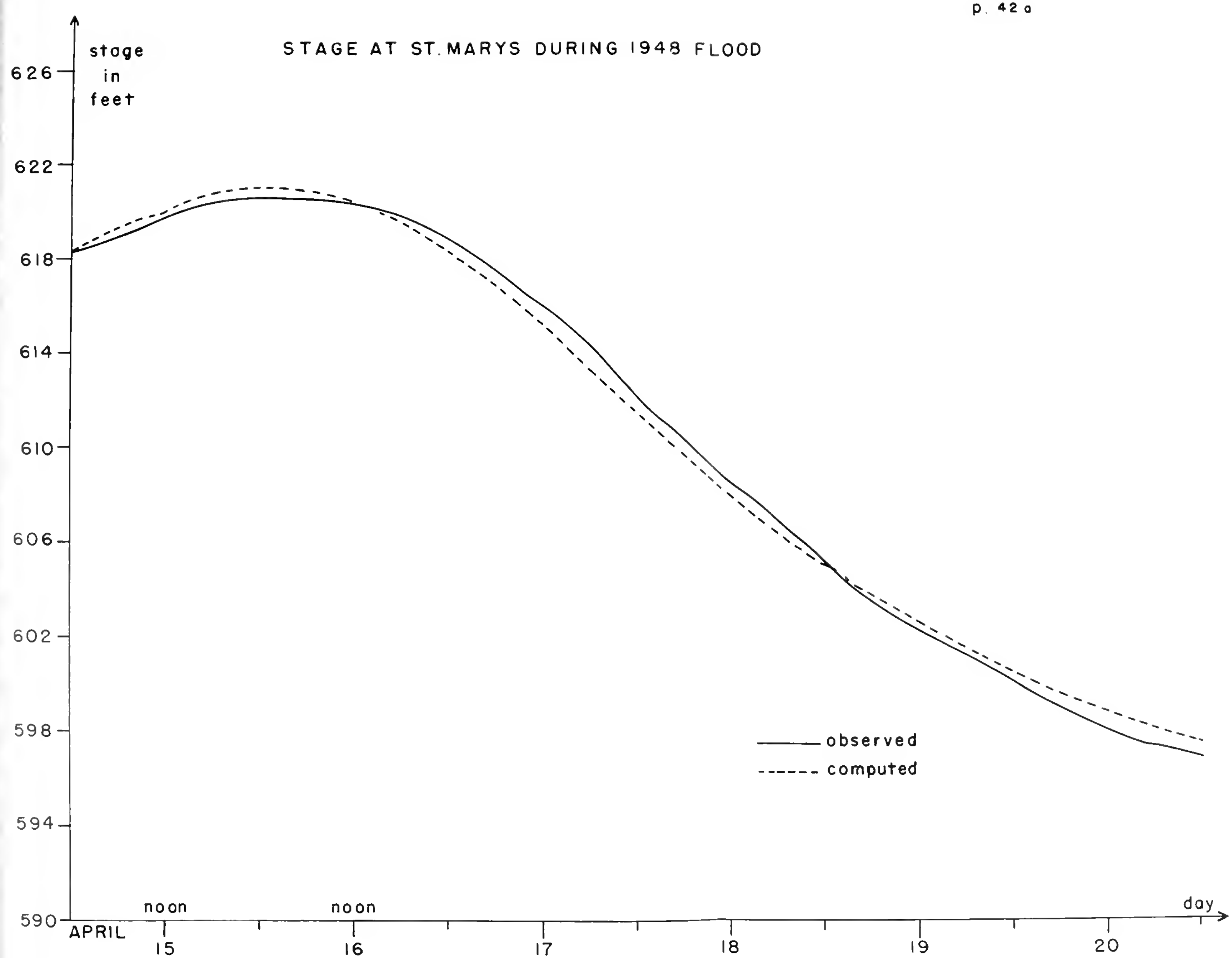
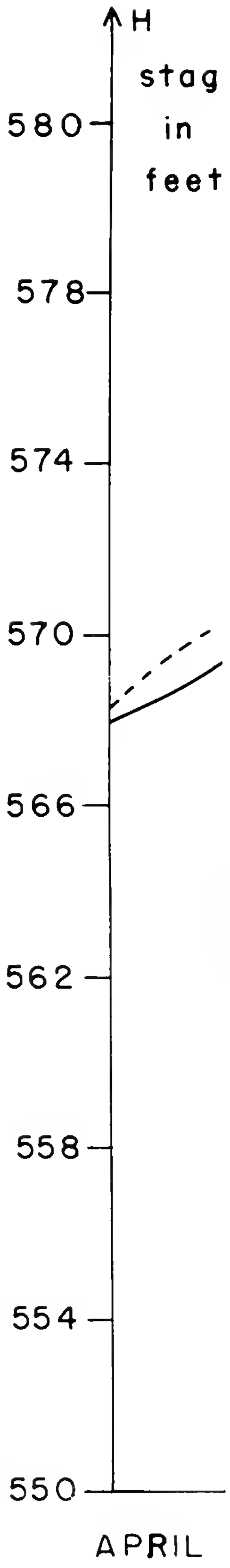

STAGE AT ST. MARYS DURING 1948 FLOOD
stage in feet
626
622
618
614
610
606
602
598
594
590
APRIL
noon
noon
15
16
17
18
19
20
day
observed
computed

Fig. 4.6a



H
stag
in
feet
580
578
574
570
566
562
558
554
550
APRIL

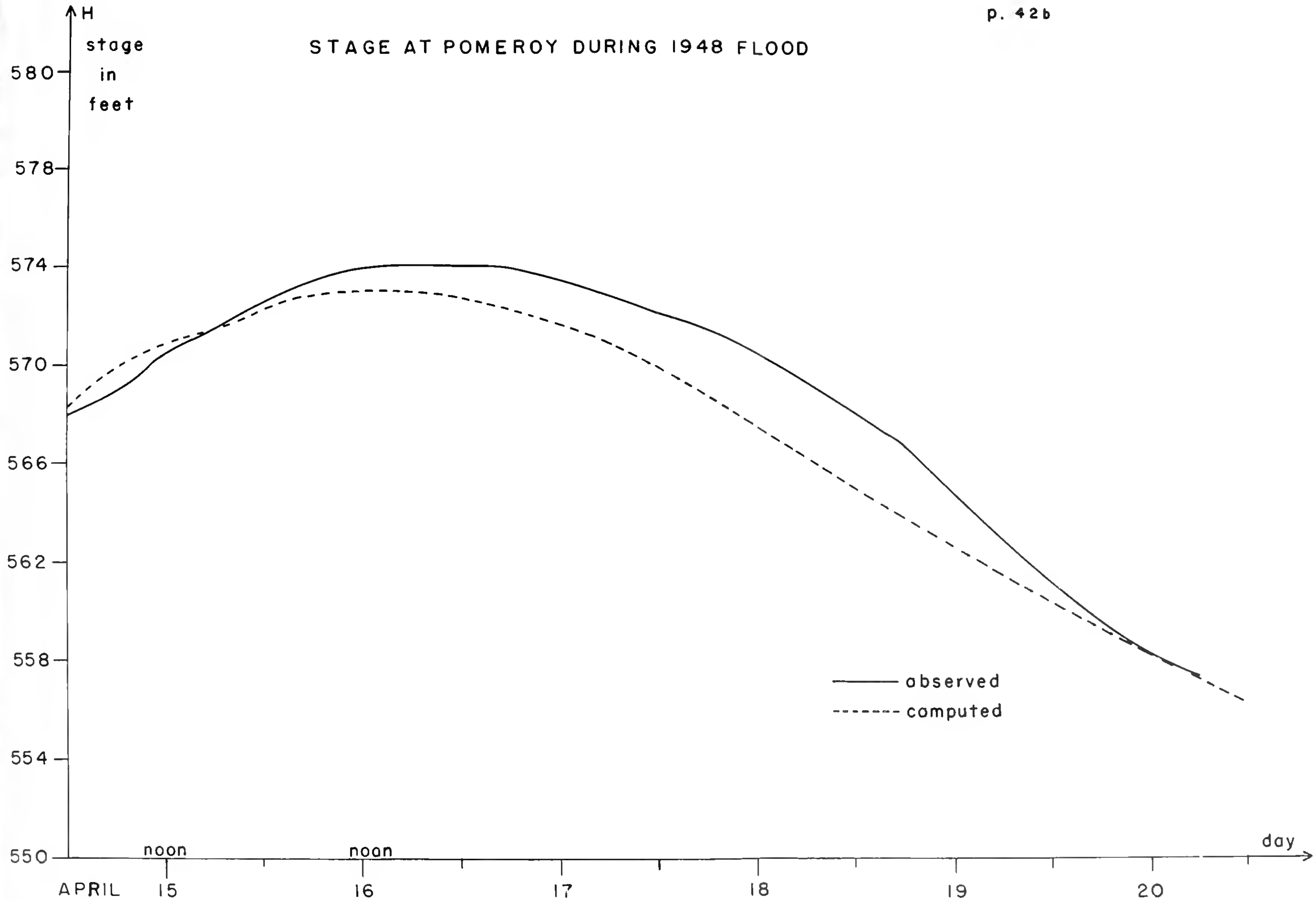
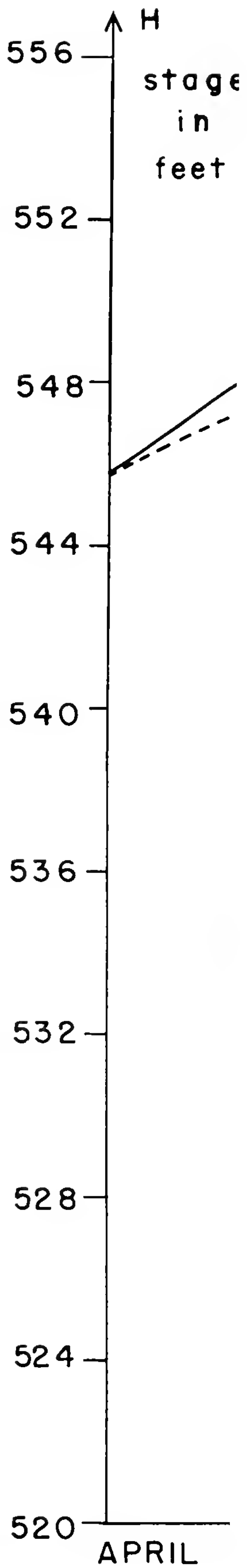

STAGE AT POMEROY DURING 1948 FLOOD
H
stage
in
feet
580
578
574
570
566
562
558
554
550
noon
noon
day
APRIL
15
16
17
18
19
20
observed
computed

Fig. 4.6b



H
stage in feet
556
552
548
544
540
536
532
528
524
520
APRIL

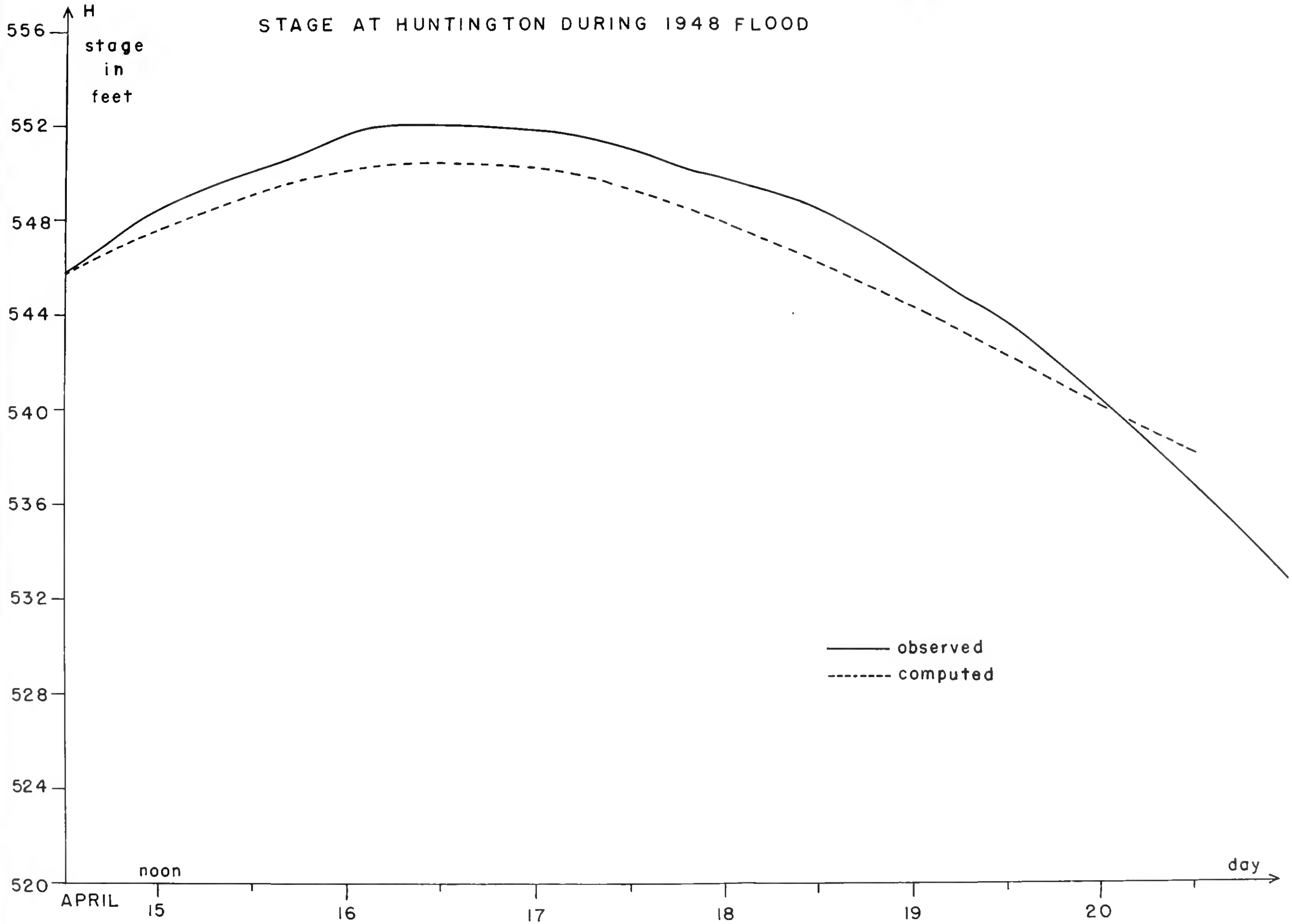
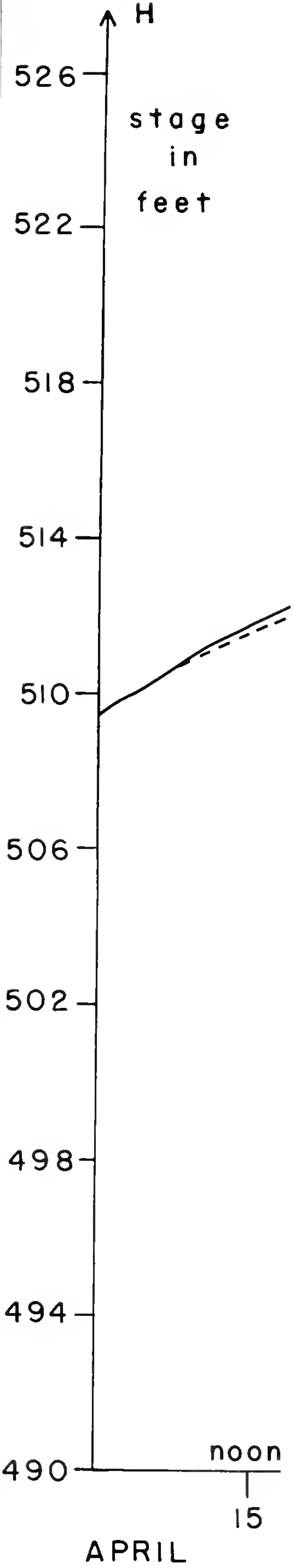

STAGE AT HUNTINGTON DURING 1948 FLOOD
H
stage
in
feet
556
552
548
544
540
536
532
528
524
520
noon
APRIL
15
16
17
18
19
20
day
observed
computed

Fig. 4.6c



H
stage in feet
526
522
518
514
510
506
502
498
494
490
noon
15
APRIL

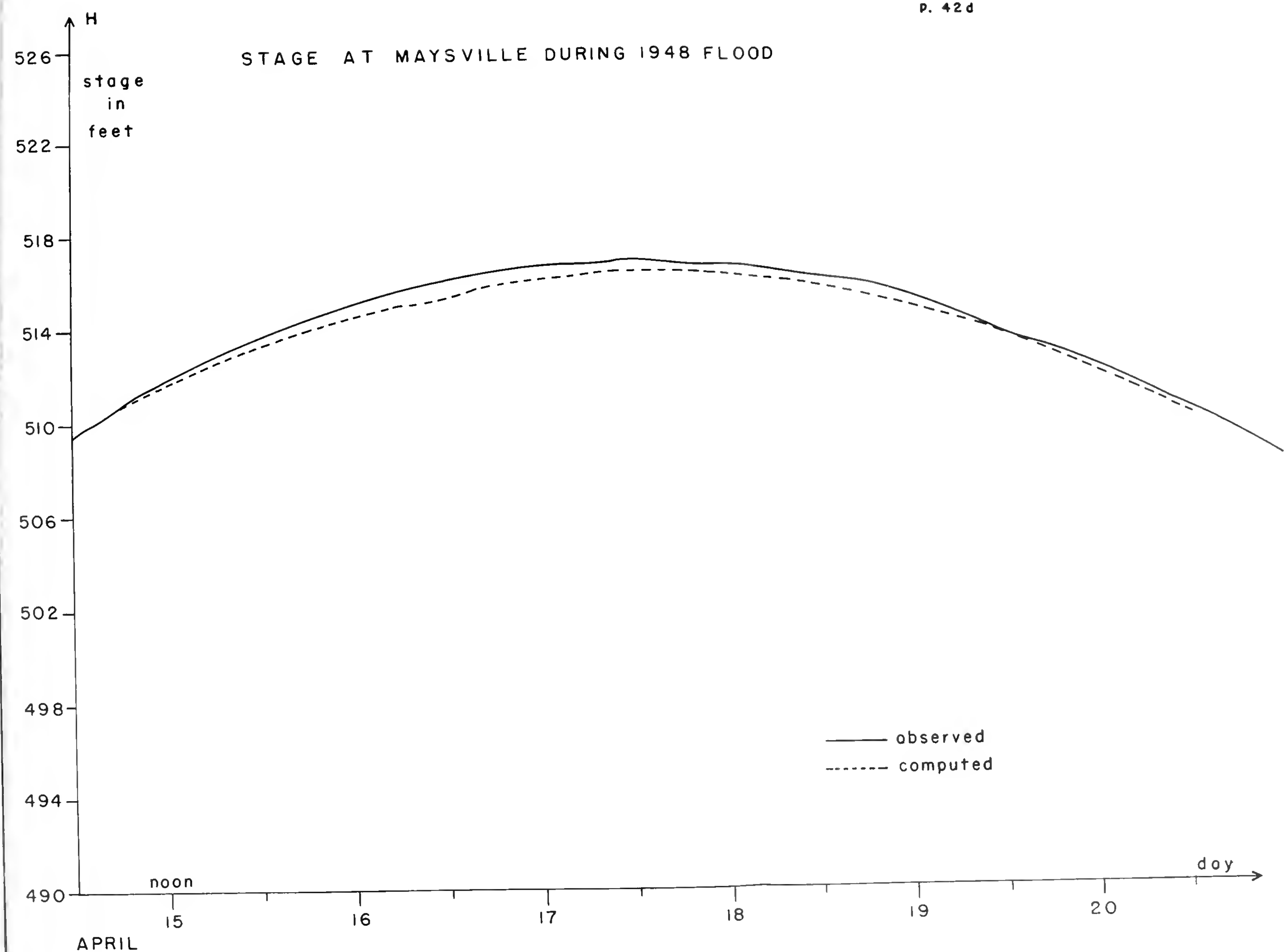
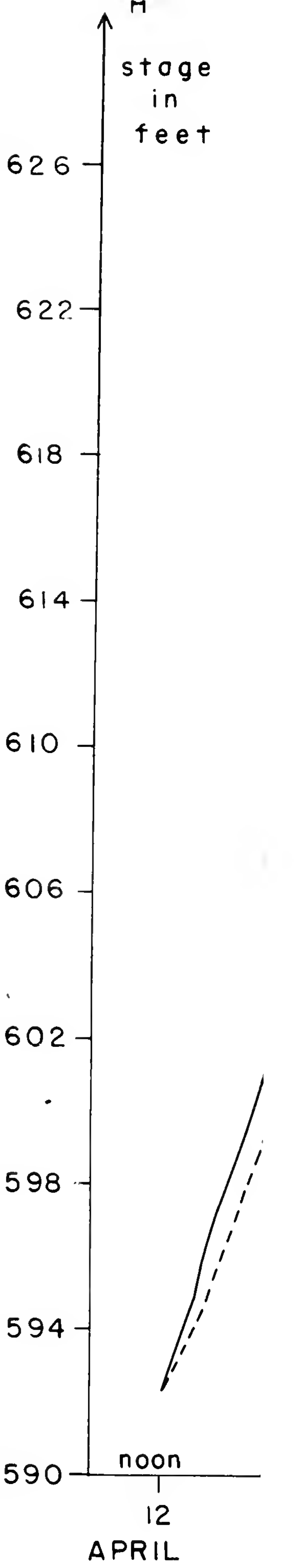

STAGE AT MAYSVILLE DURING 1948 FLOOD
H
stage in feet
526
522
518
514
510
506
502
498
494
490
noon
15
16
17
18
19
20
APRIL
day
observed
computed

Fig. 4.6d


H
stage
in
feet
626
622
618
614
610
606
602
598
594
590
noon
12
APRIL

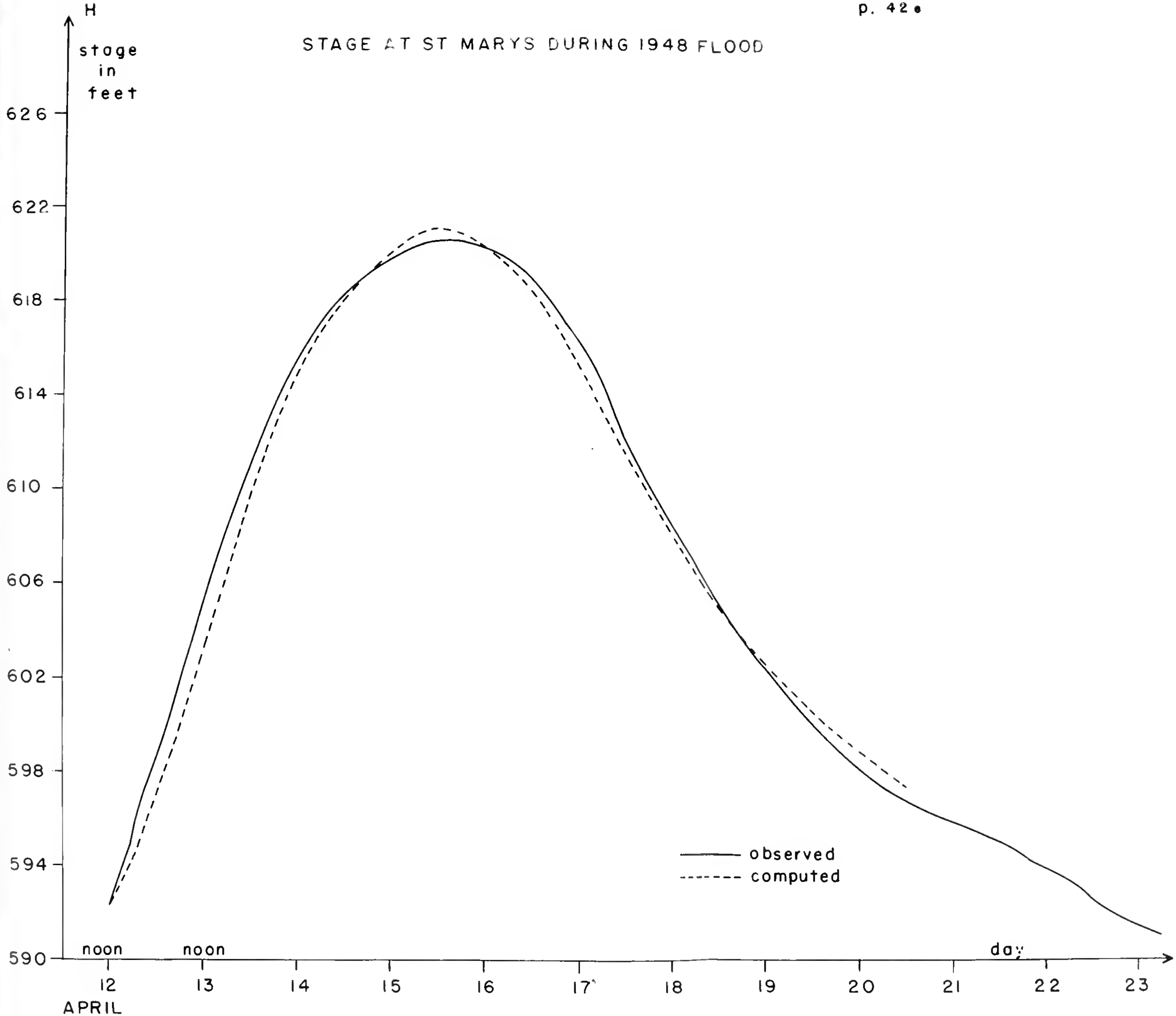

STAGE AT ST MARYS DURING 1948 FLOOD
H
stage
in
feet
626
622
618
614
610
606
602
598
594
590
noon
noon
day
12
APRIL
13
14
15
16
17
18
19
20
21
22
23
observed
computed

Fig. 4.7

and contrasts can be made. Also, in §7 suggestions of various kinds are given for improving the methods of computation, based on our experience.

So far, our discussion of results has centered around the comparison of observed with calculated stages, while velocities and discharges have received little or no attention. In general, the stage is probably the quantity of most interest, but, as we know, it cannot be computed by our methods without also computing the velocity at the same times and places. In addition, the discharge is in many cases the natural quantity to prescribe as a boundary condition. Our method requires us to replace the actual river cross sections by averages - in the case of the Ohio River, by averages over quite long distances -, and the velocities we compute are therefore also certain averages with respect to distance. In all three cases treated by us the stages obtained from our model of the actual rivers, or the reservoir, could be taken as the stages in the actual river with good accuracy; but if one were to calculate discharges by multiplying the cross section areas used by us by the velocity computed by us, the result would often differ very greatly from the observed discharge at a given point. In other words, it is necessary, in calculating discharges, to make a supplementary calculation in order to pass back from our model to the actual river. We proceed to give a simple way to obtain correct discharges.

We know that the velocity at a particular place at a given time is in general given quite accurately by the formula (cf. §3):

$$Gv^2 = gH_x \quad . \tag{4.1}$$

However, that will be true only if G, the resistance coefficient, has the correct <u>local</u> value. We have used the formula in order to compute an <u>average</u> value for G over the various reaches. How widely the average value departs from the local value at the gaging stations can be seen from

Fig. 4.8, which shows the average values of G for all of the reaches together with local values at Wheeling and Maysville. The local values were obtained from the local values of the slope H_x (these in turn were known from the basic data, which included stage measurements at two points at the ends of the reaches which were only a mile or so apart), and the local value of the velocity (from Q/A, with A the actual cross section area). The wide divergence of the average from the local values of G is doubtlessly the result of the fact that the gaging stations are placed at exceptionally constricted portions of the river.

Nevertheless, correct values for the actual local discharge can be obtained from our results, in the following way. Let us introduce first a few special notations for this purpose. By A_c and v_c we mean the cross section area used by us in the computation and the velocity obtained from our calculations; by A_a and v_a we mean the actual, local, values of the same quantities. This means that the discharge Q_c obtained directly from our results would be given by $Q_c = A_c v_c$, while the actual discharge Q_a is found from $Q_a = A_a v_a$. We introduce also the symbols G_c and G_a to distinguish between calculated (i.e. from our averages over reaches) and actual local values of the resistance coefficient. Since in all of our cases the stage H and slope H_x are correctly given on the whole by our calculations, it follows from formula (4.1) written down above that the following relation holds:

$$G_c v_c^2 = G_a v_a^2 \quad . \tag{4.2}$$

Since $Q_c = A_c v_c$ and $Q_a = A_a v_a$, the following relation for Q_a therefore holds:

$$Q_a = \frac{A_a}{A_c} \sqrt{\frac{G_c}{G_a}}\, Q_c \quad . \tag{4.3}$$

We repeat the significance of the terms in this equation: A_a and G_a are the actual local values of area and resistance

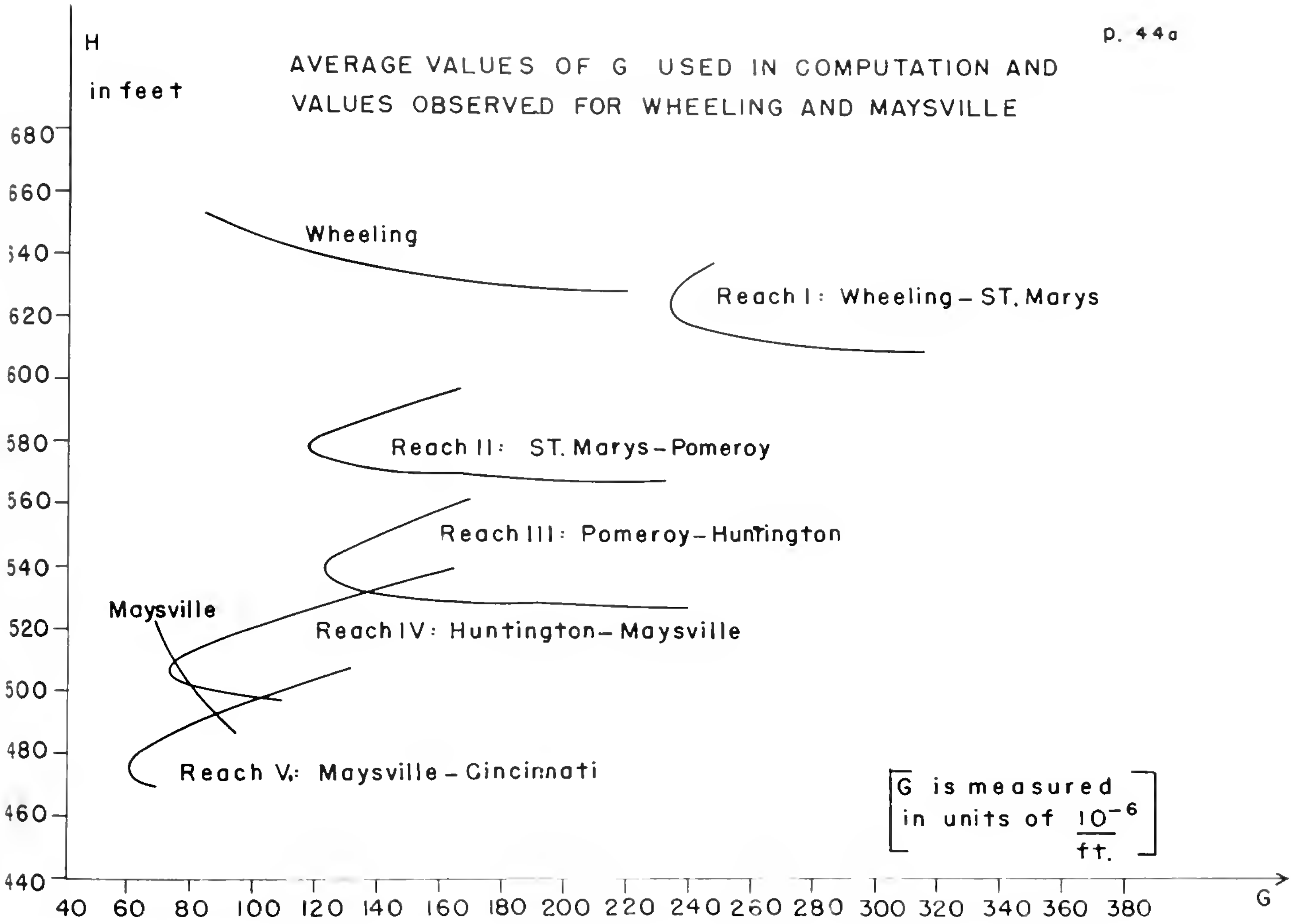

H
in feet
AVERAGE VALUES OF G USED IN COMPUTATION AND
VALUES OBSERVED FOR WHEELING AND MAYSVILLE
680
660
640
620
600
580
560
540
520
500
480
460
440
Wheeling
Reach I: Wheeling – ST. Marys
Reach II: ST. Marys – Pomeroy
Reach III: Pomeroy – Huntington
Maysville
Reach IV: Huntington – Maysville
Reach V: Maysville – Cincinnati
G is measured in units of 10^-6 / ft.
40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340 360 380
G

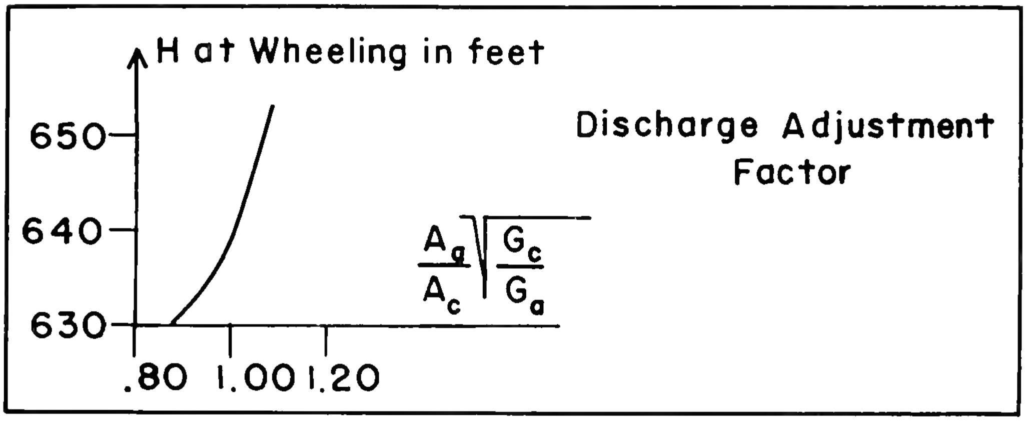
Fig. 4.8

coefficients, to be found from the basic data; while A_c and G_c are the values used in the machine computation for these quantities, and $Q_c = A_c v_c$ with v_c the value for v obtained by our computations.

Figure 4.9a, b shows the results of such a computation at Wheeling and Maysville. As one sees, the calculated discharges mirror the observed discharges very well, once the discharges are obtained by using equation (4.3). That the results would be quite wrong if Q_c were to be taken for the discharge instead of Q_a is also shown by Fig. 4.9b, which contains a graph of the factor $A_a/A_c \sqrt{G_c/G_a}$: as one sees, this factor applied to Q_c can change the discharge by as much as 100 % .

Finally, we give the result of a calculation which displays the flexibility of the numerical method. Once the basic data for the river have been coded it becomes possible to experiment in many ways with respect to hypothetical flows which would result from varying conditions in the river. As a case in point we found it very simple to carry through a computation in which the flow from one of the major tributaries of the Ohio River, the Kanawha River, was cut off for 36 hours - as might be the case if there were a dam in the Kanawha River. The upstream and downstream effects of such an operation were easily evaluated. In Fig. 4.10a, b the river stages with and without the flow from the Kanawha River are shown at the nearest gaging stations (Pomeroy upstream and Huntington downstream) from the mouth of the Kanawha River. Pomeroy is 14 miles above the tributary while Huntington is 46 miles below it. One observes that there is an upstream effect of about 1 foot with a somewhat larger effect downstream. It might be noted that the conventional flood routing procedure would in principle furnish an effect downstream only, and no effect upstream; this is a point which has been discussed at some length in the two earlier reports.

DISCHARGE AT WHEELING DURING 1945 FLOOD



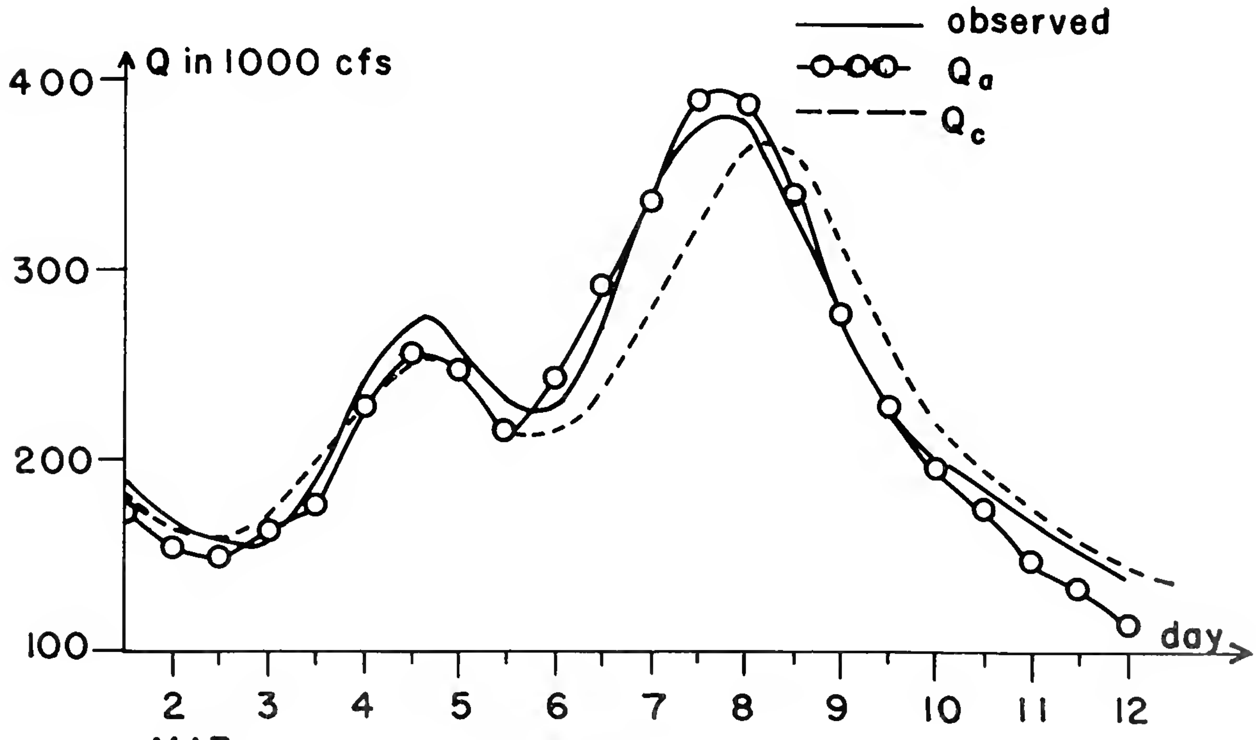


Fig. 4.9a

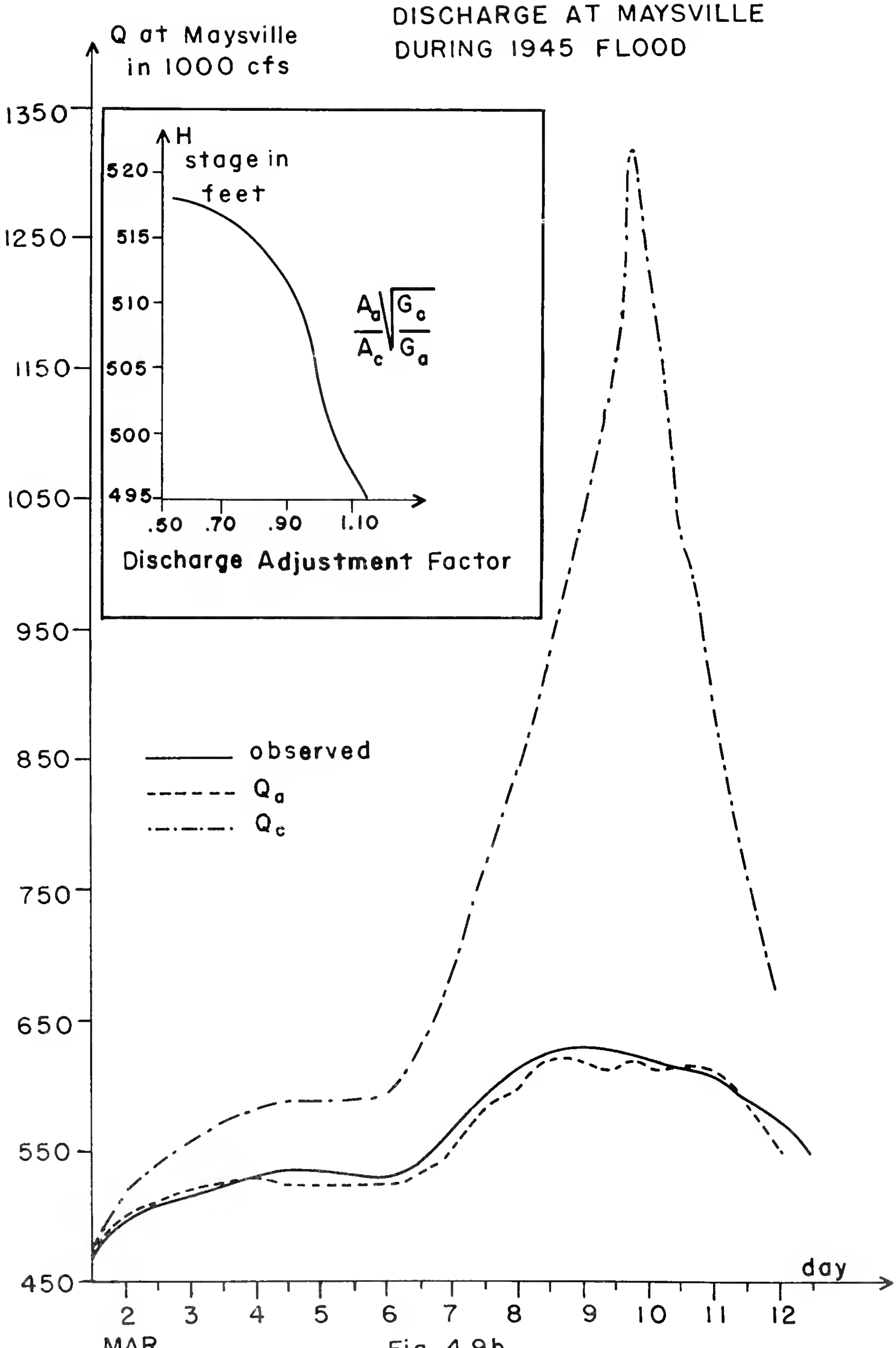

DISCHARGE AT MAYSVILLE
DURING 1945 FLOOD
Q at Maysville
in 1000 cfs
1350
1250
1150
1050
950
850
750
650
550
450
H
stage in
feet
520
515
510
505
500
495
.50
.70
.90
1.10
$\frac{A_a}{A_c}\sqrt{\frac{G_c}{G_a}}$
Discharge Adjustment Factor
observed
Q_a
Q_c
day
2
3
4
5
6
7
8
9
10
11
12
MAR

Fig. 4.9b

Calculated Stages at Pomeroy

a - with
b - without flow from Kanawha River

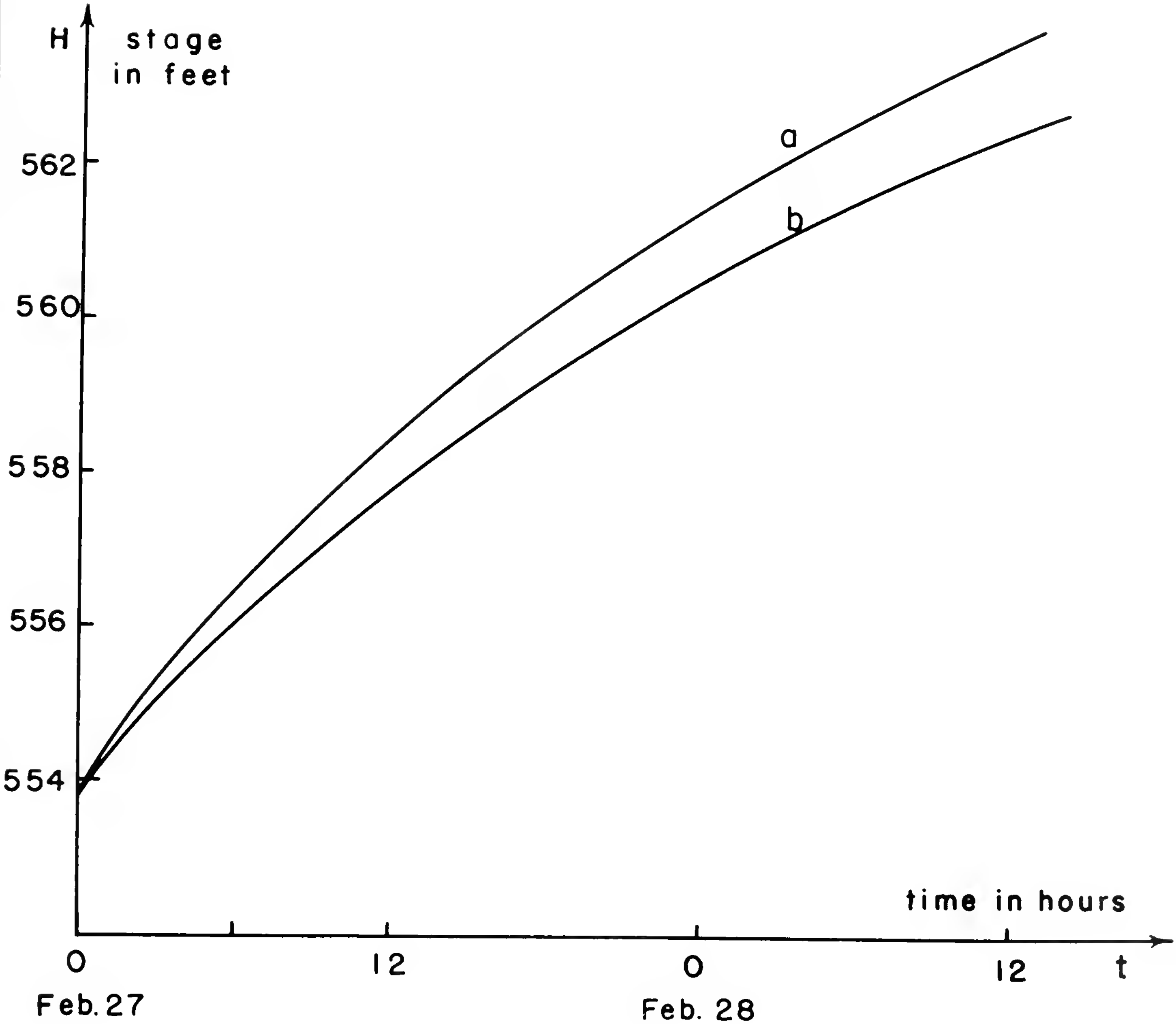


Fig. 4.10 a

Calculated Stages at Huntington
a - with
b - without flow from Kanawha River

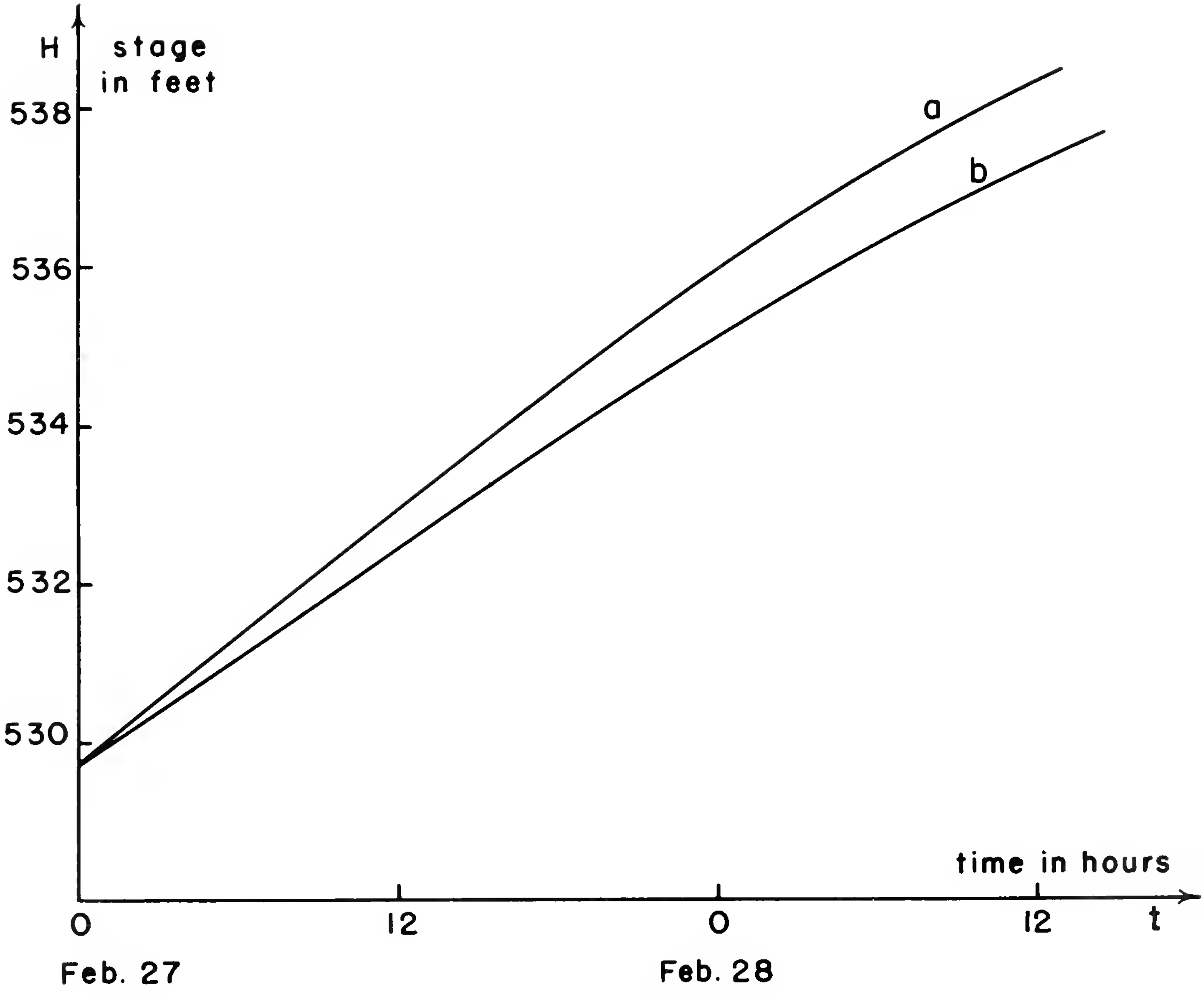


Fig. 4.10b

§5. Results for the 1947 Flood at the Junction of the Ohio and Mississippi Rivers

In the case of the junction problem the data furnished were better for our purposes than the data for the Upper Ohio River. As was noted above the data for the present case were furnished in such a way that average cross section areas over about twenty-mile stretches were known or easily calculated and the resistance coefficient was given directly over reaches somewhat shorter than those in the Upper Ohio. In fact, only slight adjustments of the area and resistance coefficients had to be made; what these were will be described shortly.

The problem to be solved involves stretches each about 40 miles in length along all three branches from Cairo. Fig. 5.1 indicates the situation schematically. As boundary conditions at Metropolis in the Ohio River and at Thebes in the Upper Mississippi River the river stages were taken from the actual records of the 1947 flood. (We performed another calculation-see second half of this section---in which the discharge data at Thebes and Metropolis were used instead of the stage data.) At Hickman in the Mississippi river below Cairo the river stages of course were available, but it was thought more reasonable to make use of an average rating curve at Hickman as a boundary condition. What this means is that the effect of the remainder of the Mississippi River below Hickman is replaced by the average relation between discharge and stage at Hickman as obtained from flood records. (Figure 5.2 displays the average rating curve at Hickman and a few of the observed points.) In addition to these boundary conditions it is necessary to impose appropriate conditions at Cairo, the junction. In effect the differential equations are solved for each of the three branches separately in the same manner as was explained in the preceding section for the Upper Ohio River; however it is necessary to piece together the three separate solutions in the branches at Cairo by making use of the appropriate continuity conditions. Reasonable

THE JUNCTION PROBLEM

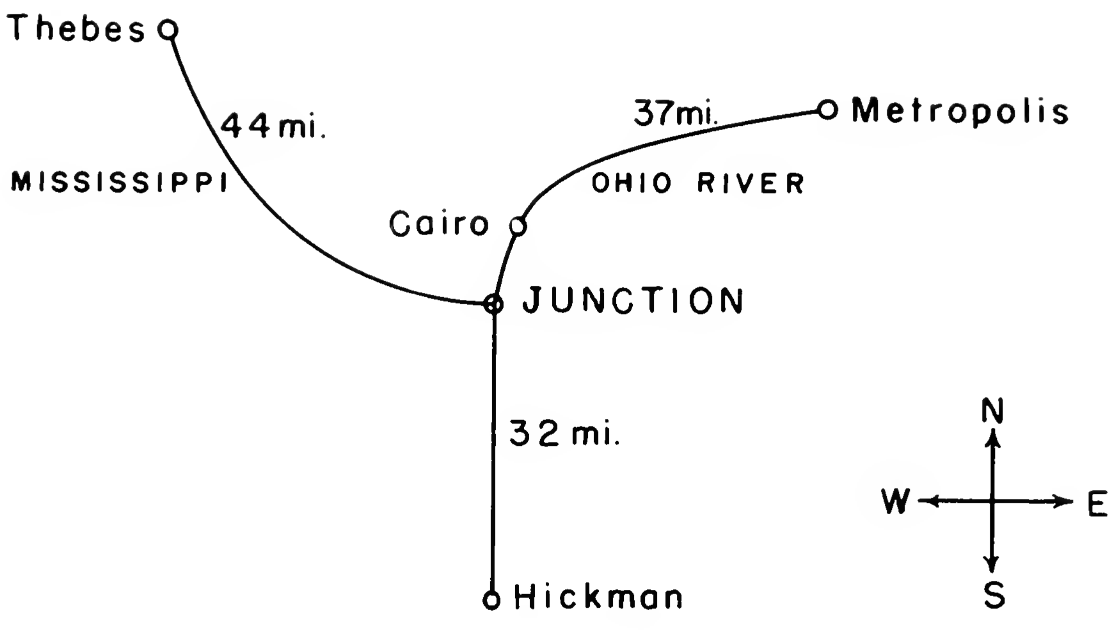


Fig. 5.1

Rating Curve at Hickman

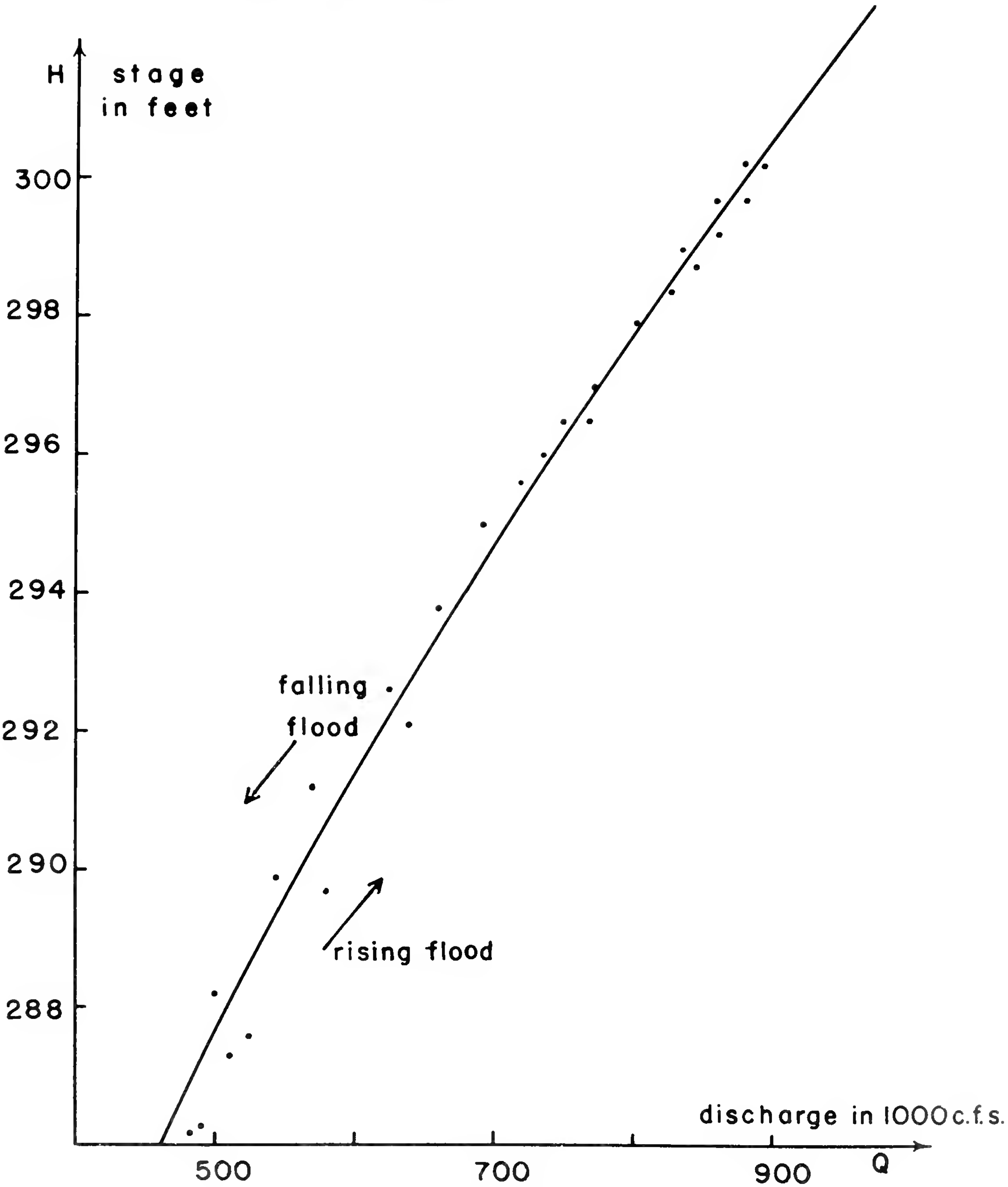


Figure 5.2

conditions at the junction are that the stage at Cairo should be the same in all three branches and that the inflow from the upper branches should equal the flow into the Lower Mississippi. These conditions together with the river stages and discharges at all of the net points at the time $t = 0$ (which corresponds to January 15,1947) determine the flow uniquely in all three branches. The results of predictions over a 16 day period are shown in Fig. 5.3. These graphs furnish the stages as functions of time at Cairo and at Hickman. As one sees even the minor irregularities in the observed stages are followed closely by the predictions up to the crest of the flood. The error was in fact never greater than 0.6 of a foot. The amount of computing time required on the UNIVAC for the 16 day prediction was less than three hours.

It is worthwhile to repeat that the stage at Hickman in the Lower Mississippi River was obtained as a consequence of assuming that the part of the Mississippi River below Hickman could be replaced by an average rating curve, that is, by an average discharge-stage relation, at Hickman. The accurate results obtained for the problem in this way indicate that a large river system could be treated in this manner. In other words even if machines with a smaller storage capacity than the UNIVAC were to be used it might still be possible to deal with lengthy stretches of a given river or river system by breaking it up into pieces of sufficiently small lengths. The total amount of machine time needed in computation is proportional to the length of the stretch and is not materially increased by the above described subdivision.

It was thought advisable to carry out the solution of the junction problem for the same flood of 1947, but to make use of the known discharges into the upper ends of the Ohio and Mississippi as a means of fixing boundary conditions, rather than to use the observed stages at these points: in practice, it would be the discharge which would be known or estimated rather than the stage. At Hickman, in the lower Mississippi

STAGES IN JUNCTION PROBLEM DURING 1947 FLOOD

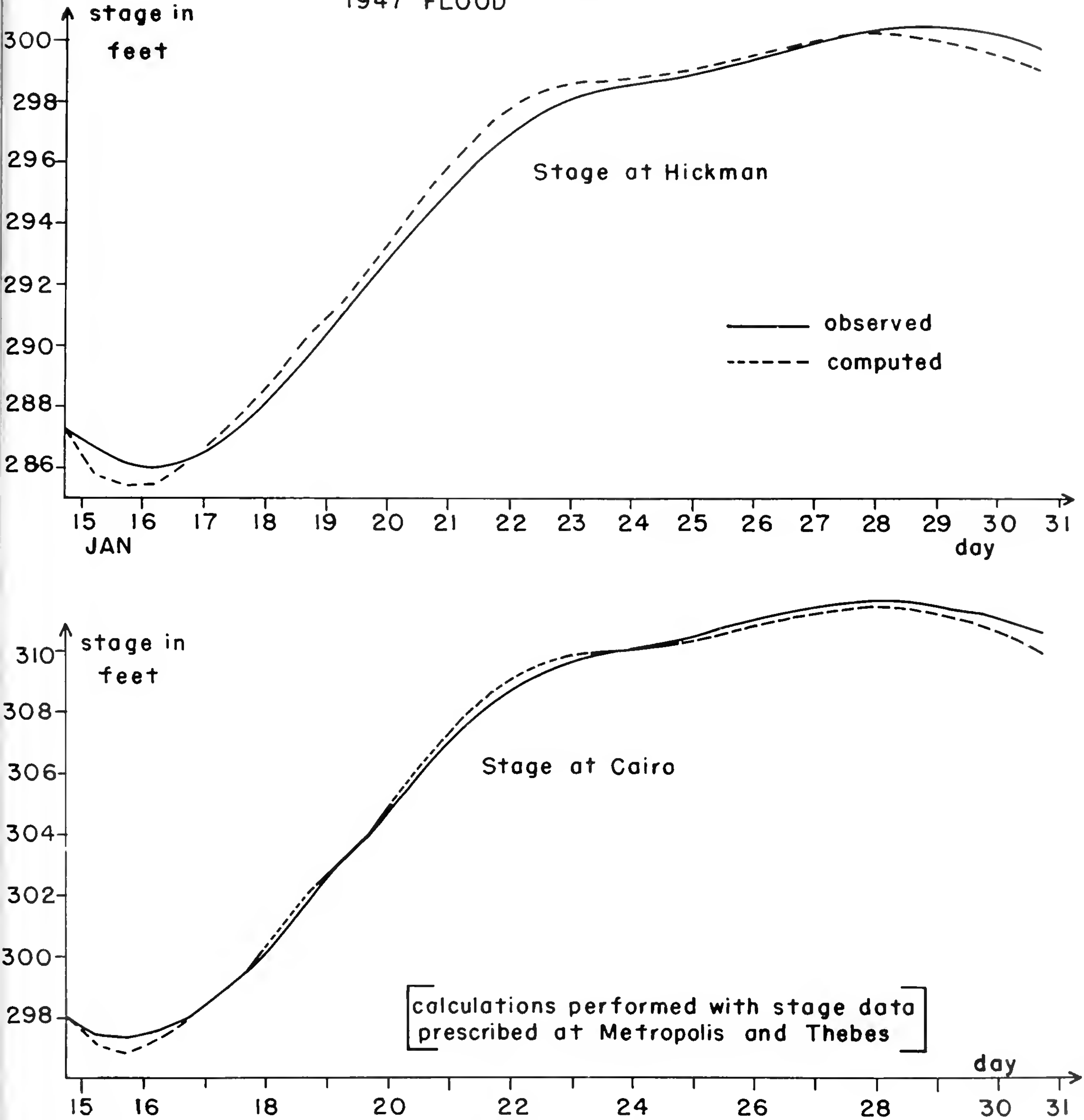


Fig. 5.3

the same average rating curve, shown in Fig. 5.2, was used as for the earlier calculation. It was found that slight adjustments in area and resistance coefficients were needed near the upper ends of the two branches above Cairo in order to reproduce the observed stages at these points (i.e. at Thebes and Metropolis) when the discharges are prescribed rather than the stages. The results with the new coefficients are very accurate, as one sees from Fig. 5.4 which presents graphs shwoing stages at Metropolis, Thebes, Cairo, and Hickman. Even the minor variations in stage are reproduced faithfully. (It is seen that with the coefficients used in the latest calculation, the stages at Thebes and Metropolis are reproduced faithfully. Therefore we would expect equally good results if the stage upstream were used as the boundary data.)

It has been stated that proper continuity conditions must be imposed at Cairo, the junction of the rivers, in order to formulate the flow problem completely. We chose, rather naturally, the conditions that the stage should be the same in all three branches, and that inflows and outflows should balance at the junction. This, however, results in a violation of the law of conservation of momentum, which is not very large, but does exist. It might be more reasonable to impose the laws of conservation of mass and momentum, and put up with a slight discontinuity in stage.* For example, one could impose the two conservation laws, and require in addition that the stages should be the same in the two upper branches, with a slight discontinuity in stage in the lower Mississippi as a result. For example, in the model of the junction problem which was treated in Report II, a discontinuity of 0.2 feet in stage would result by making the calculation in the manner suggested here.

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* There would also be a slight violation of the law of conservation of energy, in that a loss in energy would occur. Such a loss of energy at the junction might be regarded as reasonable.

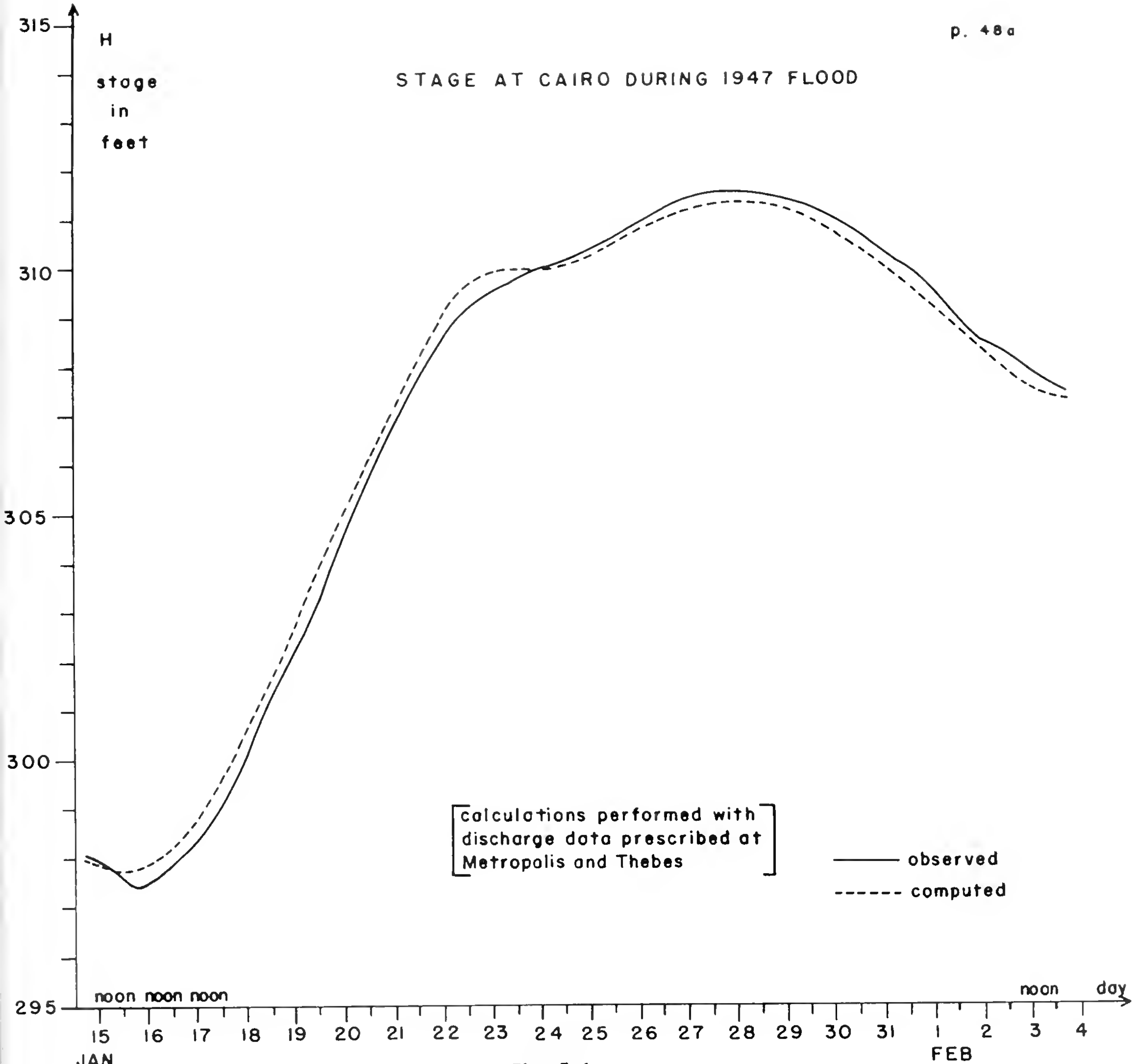

H
stage
in
feet
315
310
305
300
295
STAGE AT CAIRO DURING 1947 FLOOD
calculations performed with discharge data prescribed at Metropolis and Thebes
observed
computed
noon noon noon
noon
day
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
1
2
3
4
JAN
FEB

Fig. 5.4a

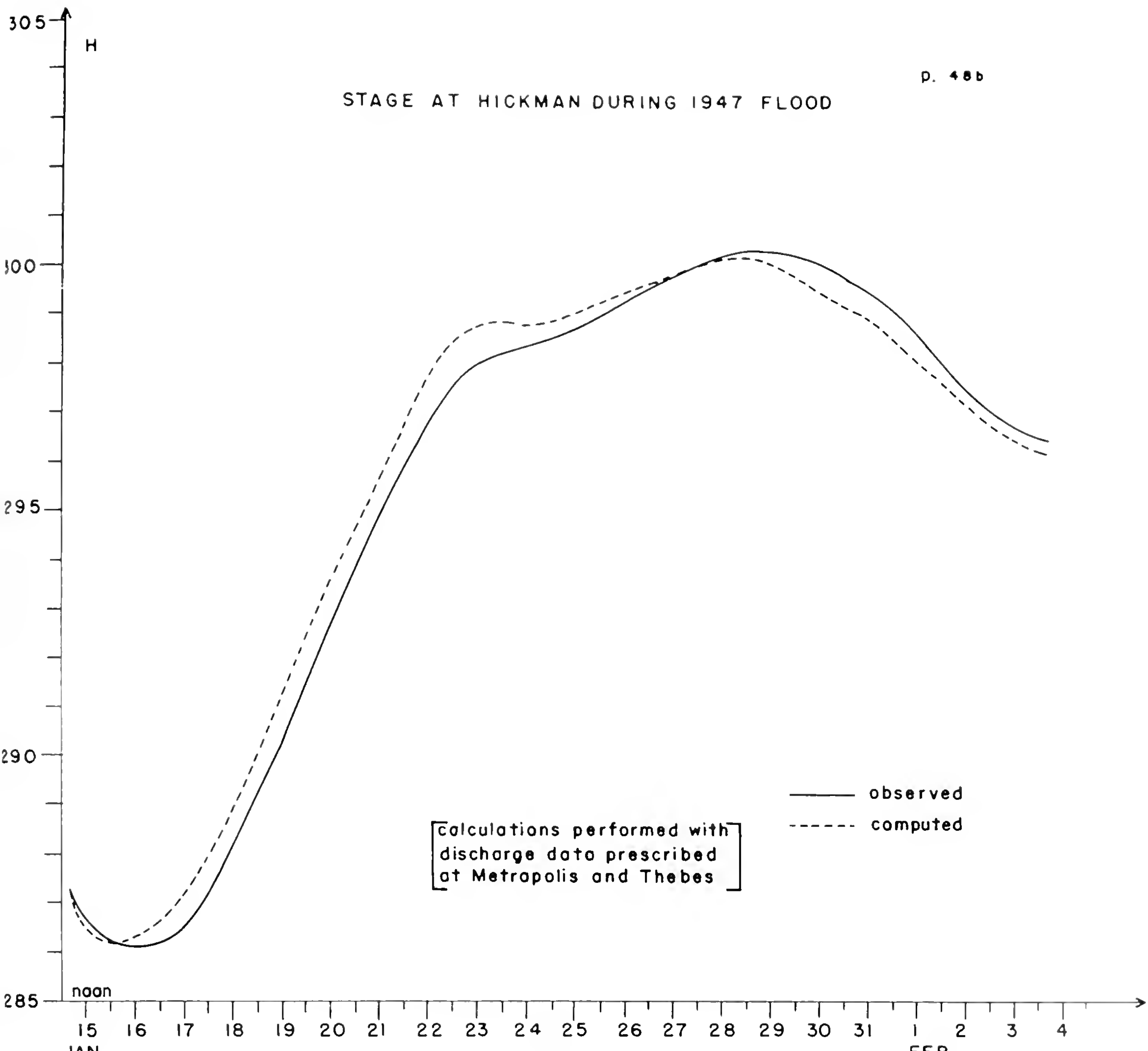

STAGE AT HICKMAN DURING 1947 FLOOD
H
305
300
295
290
285
noon
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
1
2
3
4
JAN
FEB
observed
computed
calculations performed with discharge data prescribed at Metropolis and Thebes

Fig. 5.4b

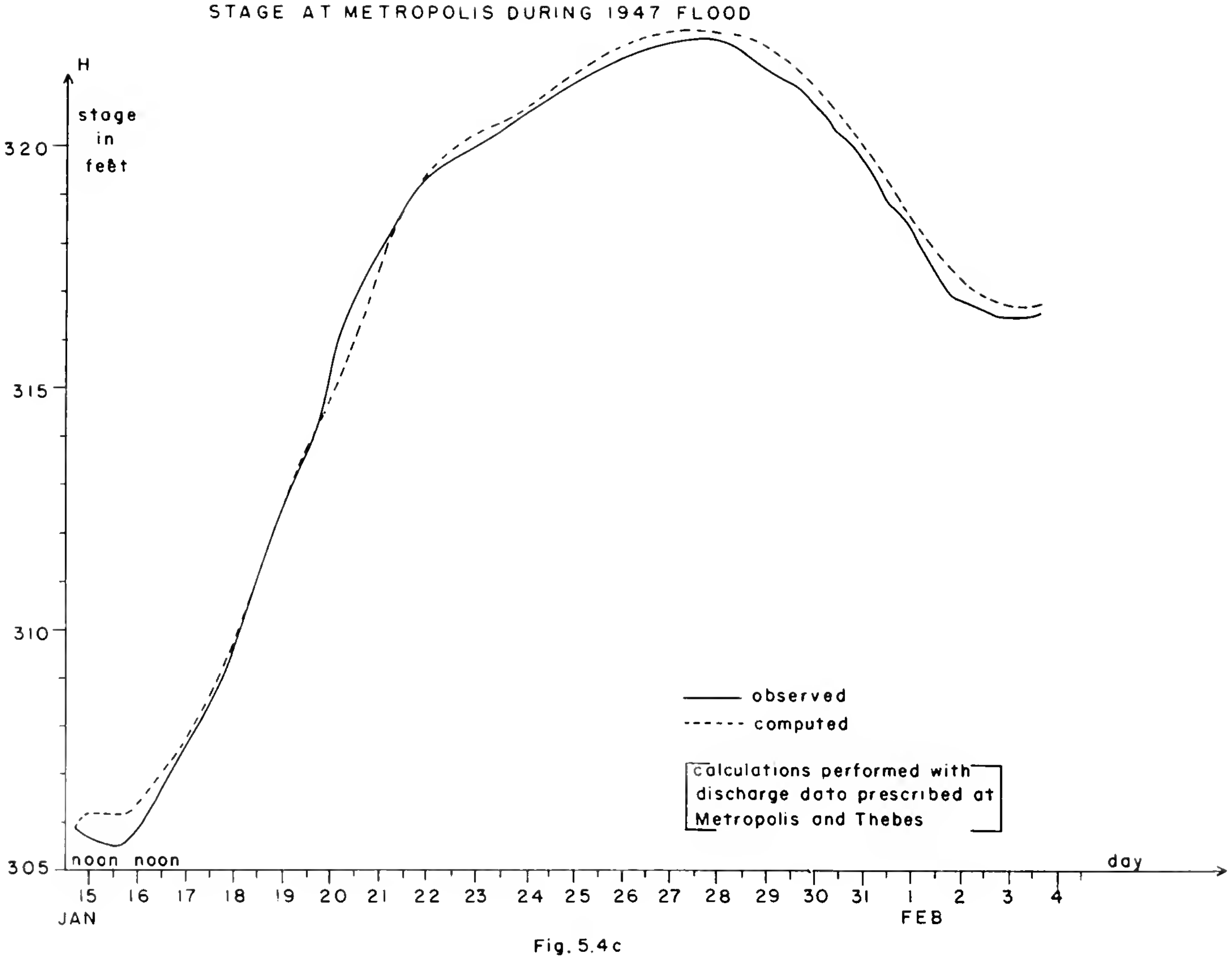

STAGE AT METROPOLIS DURING 1947 FLOOD
H
stage
in
feet
320
315
310
305
noon noon
day
15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 1 2 3 4
JAN
FEB
observed
computed
calculations performed with
discharge data prescribed at
Metropolis and Thebes

Fig. 5.4c

§6. Calculations for the 1950 and 1948 Floods in Kentucky Reservoir

Figure 6.1 is a diagrammatic sketch of Kentucky Reservoir which shows its division into reaches and names the stations where the calculated and observed stages are compared.

It has already been explained in sec. 3 how the basic data were converted into the form needed for our method of computation. We repeat here that the data were accurate for cross section areas, since storage volumes for 10-mile stretches were made available; and the resistance coefficient was essentially given directly by the basic data. That the basic data were good is evidenced by the accuracy with which the two floods were reproduced without the necessity for any but quite minor changes in the coefficients once they had been fixed from the basic data.

In sec. 3 it was already stated that the method of approximating derivatives by difference quotients that was successfully applied in the upper Ohio and in the junction problems could not be used for the flow problems in Kentucky Reservoir. The primary reason for this was that the coefficients A and B varied much too widely over 10-mile intervals, i.e. at intervals equal to the mesh width. Another contributory factor was the very rapid increase in the discharges into the reservoir from Pickwick Dam. However, the difficulty was overcome by revising the scheme of finite differences in the manner explained in sec. 3, while still retaining the mesh width of 10 miles. We proceed to discuss our results for the floods of 1950 and 1948.

In both floods the data prescribed at the two boundaries, i.e. at Pickwick Dam upstream and at Kentucky Dam downstream, consisted in the observed discharge Q as a function of time. Figure 6.2 gives the discharges for the 1950 flood. As one sees, the discharge into the reservoir at Pickwick Dam increases very rapidly at the beginning, going from 55,000 c.f.s. on Jan. 4 to 220,000 c.f.s. between Jan 6 and 7, after which

THE KENTUCKY RESERVOIR

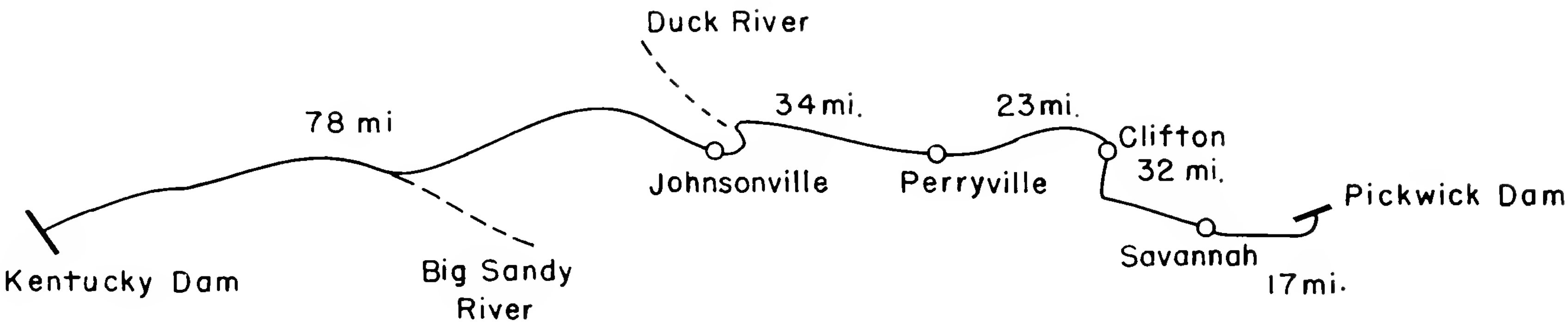
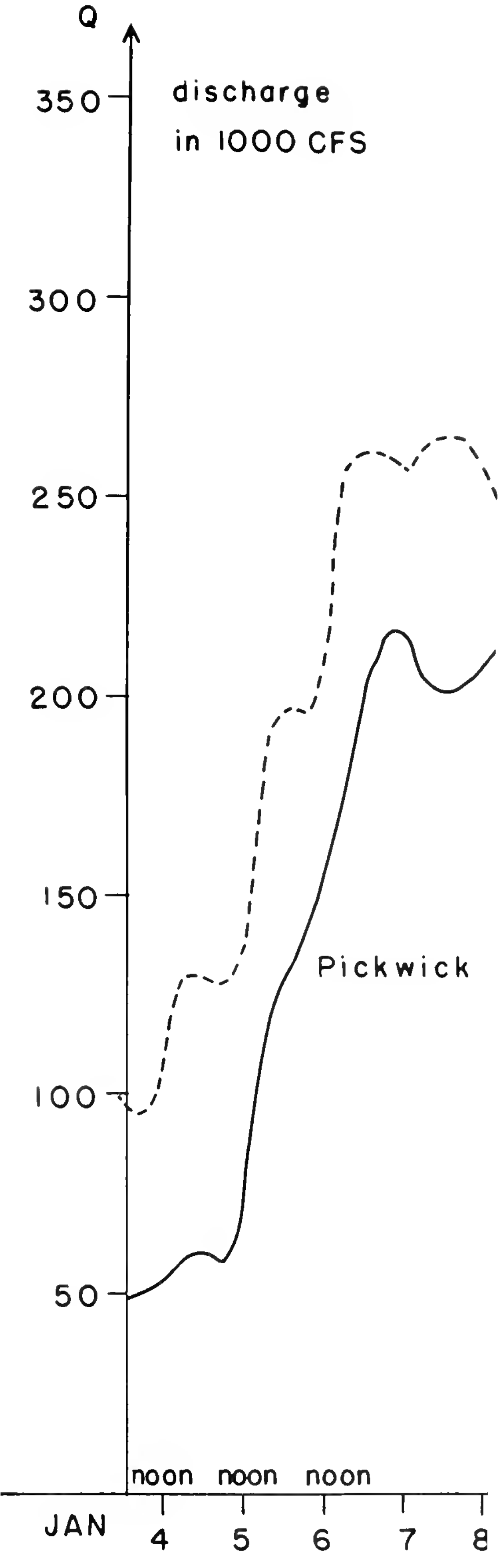


Fig. 6.1


Q
discharge
in 1000 CFS
350
300
250
200
150
100
50
Pickwick
noon
noon
noon
JAN
4
5
6
7
8

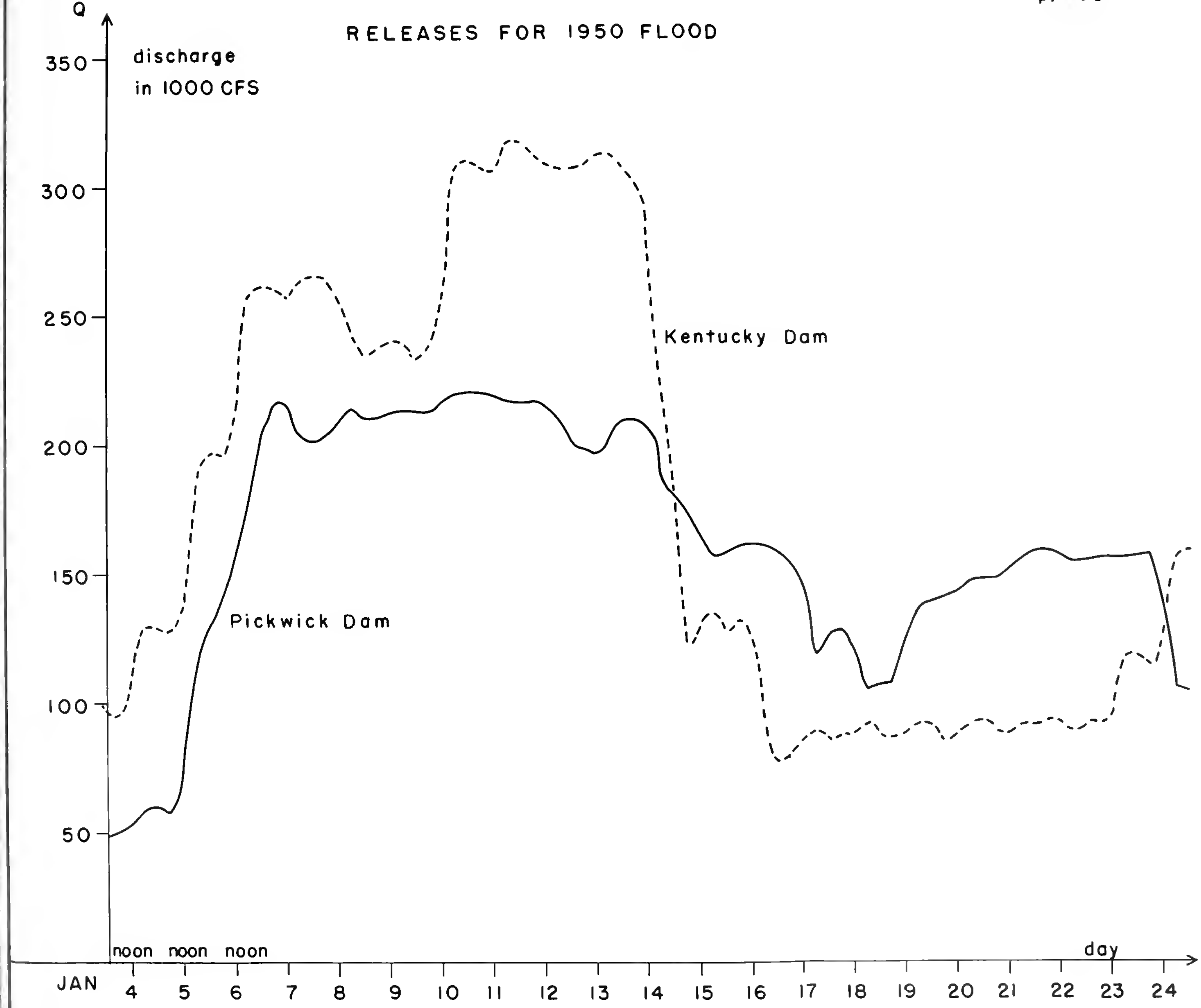
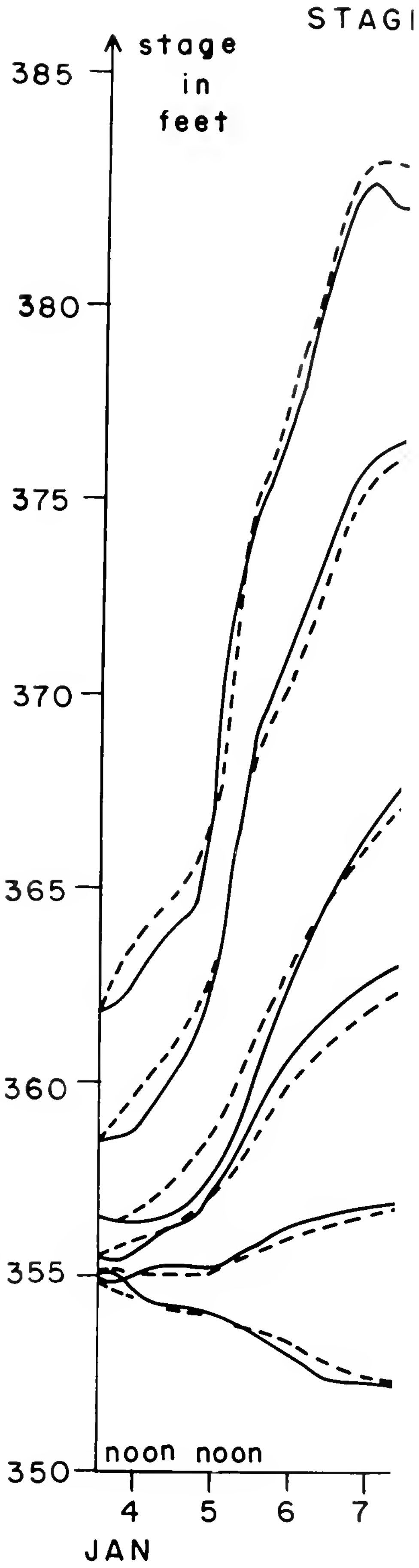

RELEASES FOR 1950 FLOOD
Q
discharge
in 1000 CFS
350
300
250
200
150
100
50
Kentucky Dam
Pickwick Dam
noon noon noon
day
JAN
4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Fig. 6.2

the discharge remains nearly stationary for about a week, after which there is a fairly sharp decrease to about 110,000 c.f.s., followed again by a slow rise to about 160,000 c.f.s. The releases of water from the reservoir at Kentucky Dam follow much the same pattern, but the changes are more abrupt and the maxima and minima are farther apart. The results of the numerical solution of the flow problem in the reservoir are given in Fig. 6.3, which shows the stages at various points along the reservoir as a function of time. It should be remarked that the stages at the two dams at the ends of the reservoir, as well as those at intermediate points, were obtained by calculation from the difference equations. As one sees, the observed flood is reproduced for 21 days very accurately at most of the stations. In fact, with the exception of Savannah and Clifton, where differences of 1 to 1.5 feet occurred, the errors are of the order of inches. Upon comparing the releases at the dams, as given by Fig. 6.2, with the resulting stages given by Fig. 6.3, we observe that the large releases into the reservoir resulted, as they should, in rapid increases in stage at the upper end of the reservoir. At Perryville, at about the midpoint of the reservoir, the stage increased only slightly, while it decreased at Kentucky Dam, the lower end of the reservoir. This resulted because of the high releases of water at Kentucky Dam. The relatively smooth curve giving the stage at Kentucky Dam should be compared with the rather wildly fluctuating curve for the discharge at the same point. In fact, the discharges (and hence also the velocities) through most of the reservoir in general vary much more rapidly than the stages. Another comment is also of interest in this connection. One sees that very heavy <u>releases</u> at Kentucky Dam seem not to affect the accuracy of the stage calculations at this point; the reason is that such releases cause a receding wave which lowers the stage and tends to smooth out as it propagates. On the other hand, rapid inflows at Pickwick do affect the accuracy of the approximate solutions,



STAGI
stage in feet
385
380
375
370
365
360
355
350
noon noon
4
5
6
7
JAN

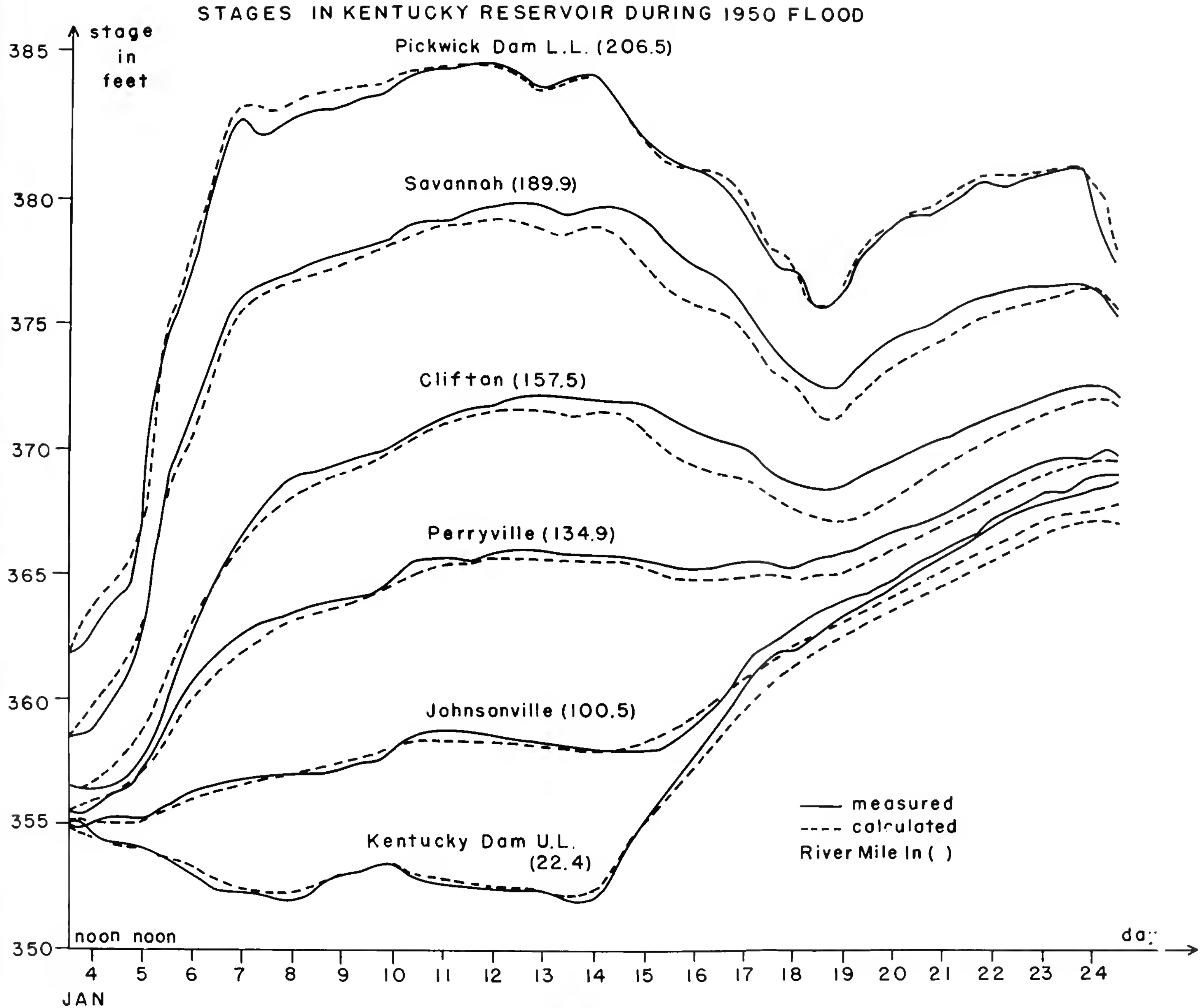

STAGES IN KENTUCKY RESERVOIR DURING 1950 FLOOD
stage in feet
385
380
375
370
365
360
355
350
Pickwick Dam L.L. (206.5)
Savannah (189.9)
Clifton (157.5)
Perryville (134.9)
Johnsonville (100.5)
Kentucky Dam U.L. (22.4)
measured
calculated
River Mile in ( )
noon noon
day
4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
JAN

Fig. 6.3

since rising stages are accompanied by steepening of wave profiles, and the finite difference approximations based on fixed mesh widths become less accurate. This may very well be the cause of the inaccuracies at Savannah and Clifton toward the end of the flow period: the effect of the long-continued high rate of discharge at Pickwick Dam has perhaps led to inaccuracies which might be corrected by using a finer mesh width or by going over a more refined basic scheme of calculation.

In Figs. 6.4 and 6.5 the results of calculations for the 1948 flood are shown. The calculated results were obtained using the same coefficients as were used to calculate the flow in the 1950 flood. The results are given for 7 1/4 days only, stopping at 6:00 A.M. on Feb. 13 since the flood stages increased from then on at such a rate* that our approximate method of computation of finite differences was very inaccurate - in fact, it seemed even to diverge. A finer mesh width $\triangle x$ (and of course, a shorter time interval $\triangle t$), or else a radical revision of the basic method would be necessary in order to obtain accurate results in this case. However, for the first 7 1/4 days the check with the observed flood is, on the whole, quite good. At Kentucky Dam it is very good, and even at Pickwick Dam, where the observed and computed stages differed most, the error is not excessive. At all stations, the shape of the curve of the observed stages is accurately reproduced. The same general remarks made above for the 1950 flood hold also for the 1948 flood, which was also characterized by heavy releases at the dams closing the two ends of the reservoir.

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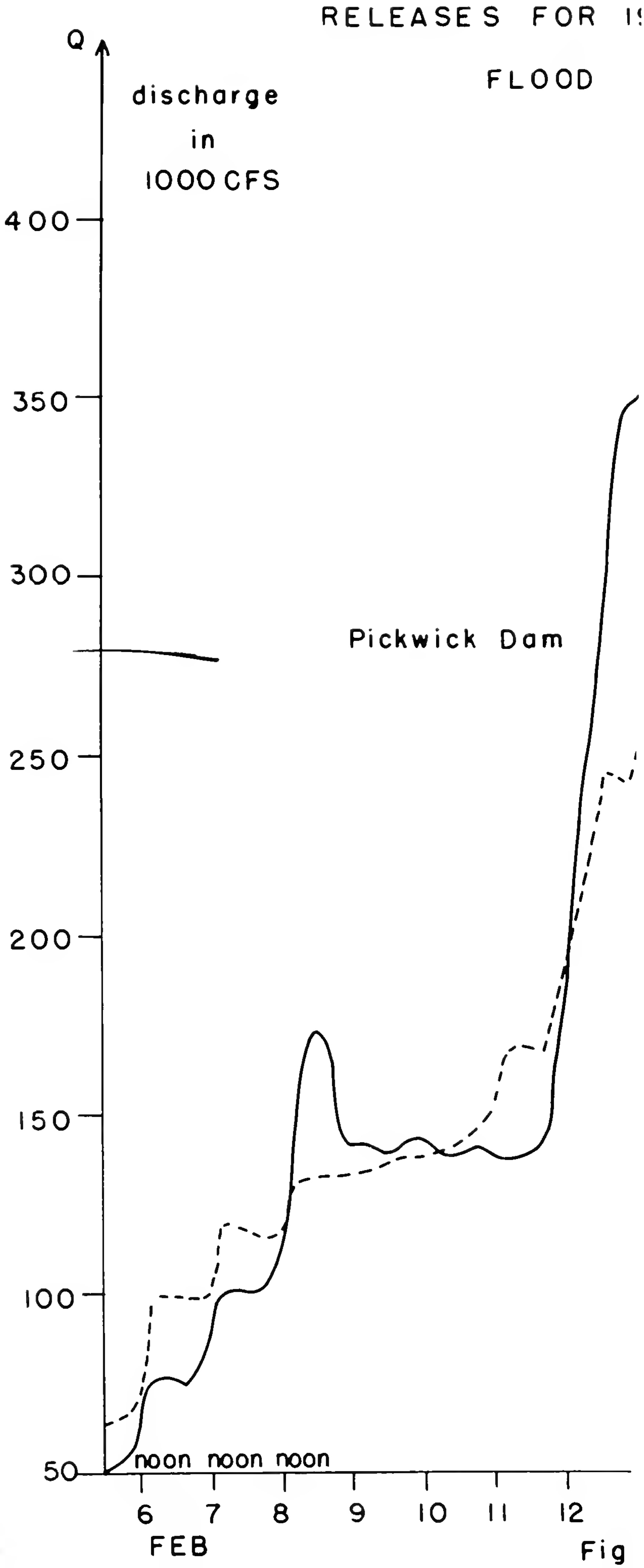
* The discharge increased at Pickwick Dam from Q = 185,000 c.f.s. to 345,000 c.f.s. within 18 hours on Feb. 12. Upon comparing Figs. 6.3 and 6.5 one observes that by Feb. 12 the stages in the 1948 flood had already reached the maxima recorded in the 1950 flood - hence further calculations for the 1948 flood could not lead to a verification of the coefficients based on the results for the 1950 flood, which was another reason for stopping at this point.



RELEASES FOR 19
FLOOD
Q
discharge
in
1000 CFS
400
350
300
250
200
150
100
50
Pickwick Dam
noon noon noon
6
7
8
9
10
11
12
FEB
Fig

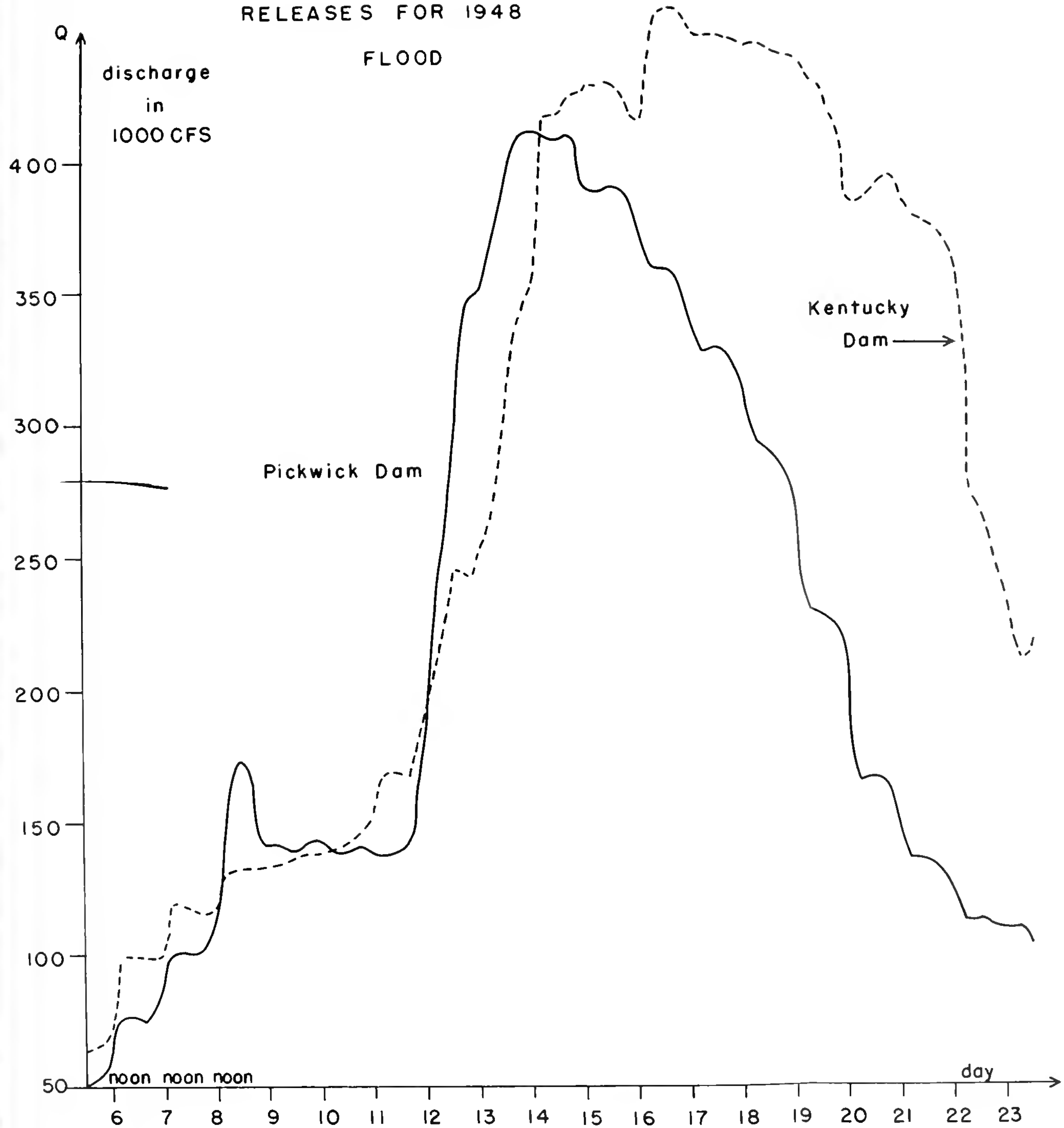
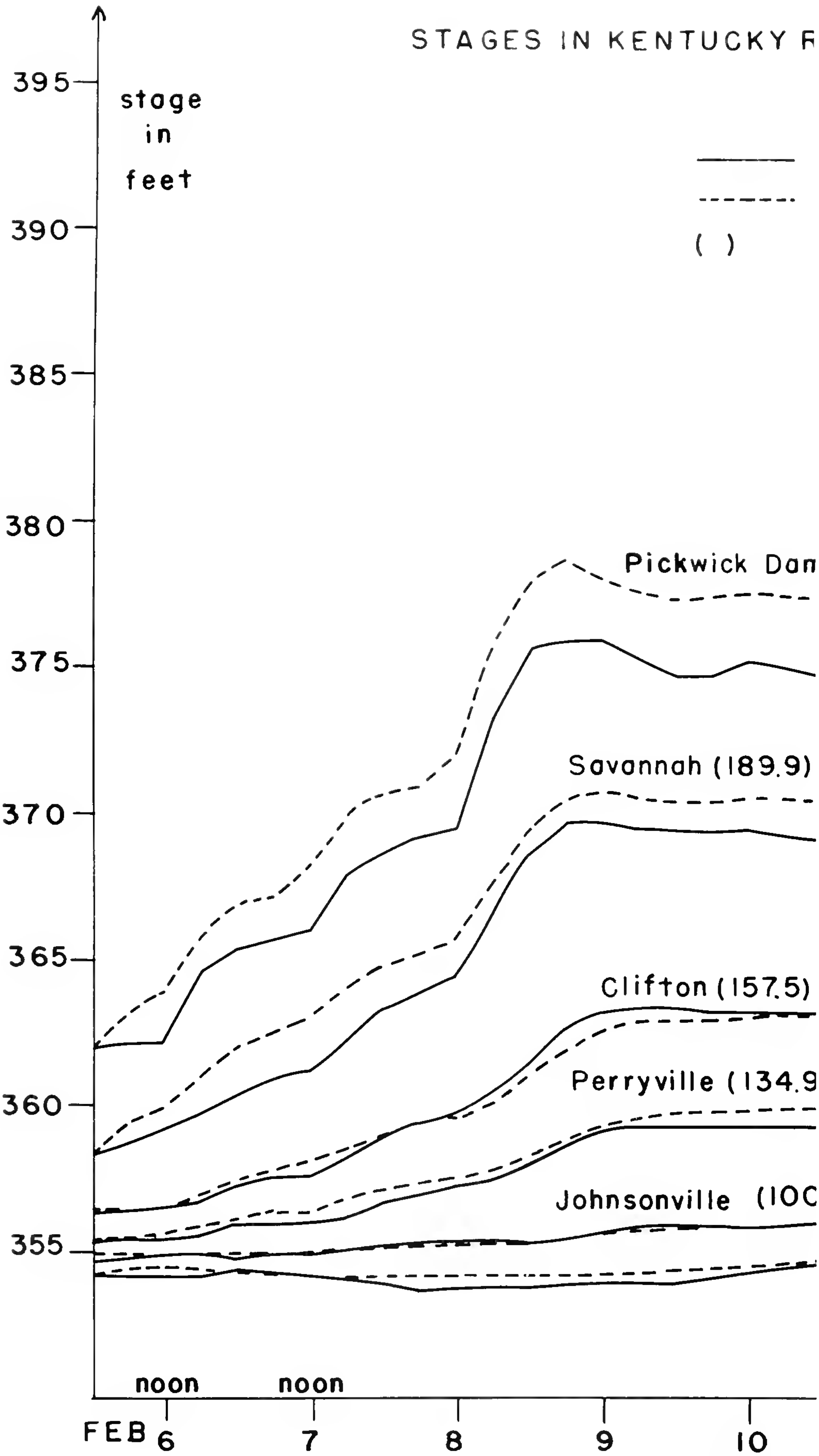

RELEASES FOR 1948
FLOOD
Q
discharge
in
1000 CFS
400
350
300
250
200
150
100
50
Pickwick Dam
Kentucky
Dam
noon noon noon
day
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
FEB

Fig. 6.4


STAGES IN KENTUCKY R
stage in feet
395
390
385
380
375
370
365
360
355
( )
Pickwick Dan
Savannah (189.9)
Clifton (157.5)
Perryville (134.9
Johnsonville (10C
noon
noon
FEB 6
7
8
9
10
Fig. 6.5

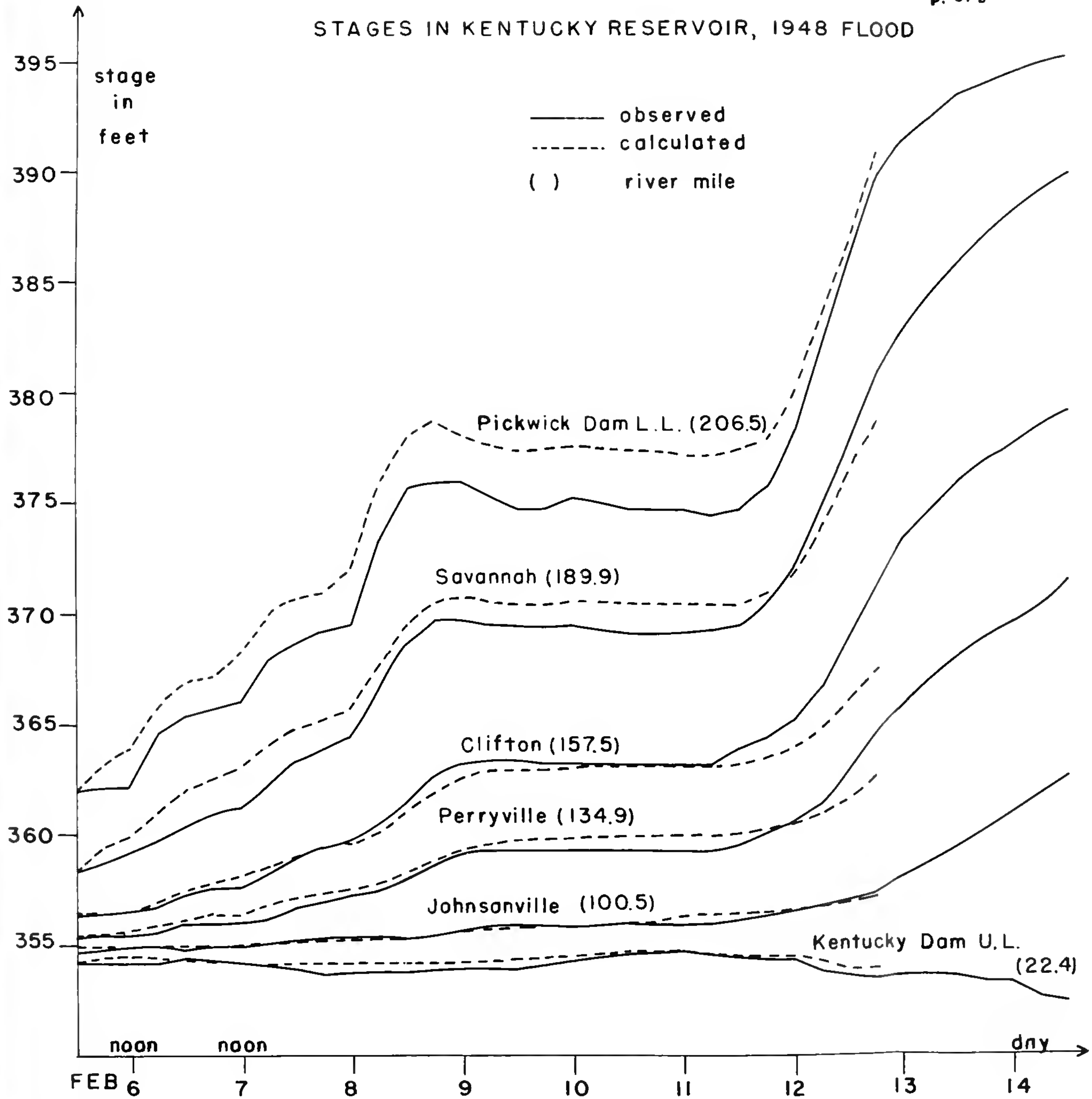

STAGES IN KENTUCKY RESERVOIR, 1948 FLOOD
stage in feet
observed
calculated
( ) river mile
Pickwick Dam L.L. (206.5)
Savannah (189.9)
Clifton (157.5)
Perryville (134.9)
Johnsonville (100.5)
Kentucky Dam U.L. (22.4)
395
390
385
380
375
370
365
360
355
noon
noon
day
FEB 6
7
8
9
10
11
12
13
14

Fig. 6.5

The results of the calculations for Kentucky Reservoir show, to our minds, that the methods used by us can be very successfully used to solve practical problems connected with the operation of large reservoirs. For example, it is now possible to make quickly and cheaply a series of calculations of flows through Kentucky Reservoir when various hypotheses are made concerning initial states in the reservoir and the discharge schedules at the two ends of the dam. By varying parameters in a systematic way such calculations would serve to furnish curves useful for operational purposes.

§7. Discussion of Factors Affecting Accuracy. Suggestions for Improvement of the Numerical Methods.

In carrying out the computations described above, insufficient time and money were available to improve the accuracy of the results in a number of cases in which there is little doubt of the possibility of improving them - often by rather obvious devices. Our aim was to use our resources of time and money in as efficient a manner as possible in order to carry out the basic task of investigating the general possibility and practicability of numerical methods for attacking flow problems in large rivers and reservoirs. As a consequence, finished results are not available in all cases, above all in the Ohio River. It would require a major effort to revise the methods used for the Ohio River* - in fact, once having coded it in the way first thought reasonable, not too radical changes were possible for us thereafter since we wished to have an opportunity to study the other two problems also. It seems therefore reasonable to devote this section of our report, even at the expense of occasional repetition, to a discussion a) of what our experience has taught us about the sources of error, and b) to offer a few possibilities for improving the numerical methods.

a) Factors affecting accuracy:

1. <u>Meshwidth Δx.</u> In general, a mesh width $2\Delta x$ = 10 miles seems reasonable in the three cases treated by us, except in Kentucky Reservoir, where the variations in the coefficients were too rapid. For the 1950 flood in the Kentucky Reservoir it was possible to use a mesh width of 10 miles, but only upon introducing a more complicated method of approximating derivatives than was necessary in the other two cases. For the 1948 flood in Kentucky Reservoir, ten miles proved to be too large a value for $2\Delta x$ once the discharges started increasing with great rapidity at Pickwick

* This was the problem with which we started. In retrospect, we feel now that it would have been better to have undertaken it last.

Dam.* Probably $\triangle x$ should be decreased by a factor of two or three, or, perhaps better, a different approximation scheme (perhaps the implicit scheme to be described below) should be employed. It should be said, however, that the computation time required if the mesh width were to be reduced by a factor of two or three would still be tolerable in Kentucky Reservoir even if the Univac were used, and if a faster machine such as the IBM 704 were to be used, the calculating time would be negligible even for these small mesh widths.

2. Ungaged local inflow. In the Ohio River the ungaged local inflow is a very important factor in the data, but is much less important in the other two problems treated by us. In the Ohio River this contribution can be quite high - in the reach St. Mary's - Pomeroy, for example, it is 40 % of the total flow, and in many cases it is as much as one and a half times the gaged inflow from tributaries. In the whole stretch from Wheeling to Cincinnati, there is one time when the discharge of the main stream at Wheeling is 160,000 cfs while 470,000 cfs flows over the banks. The accuracy with which the ungaged flow has been given is, of course, not precisely known. We suspect it to be not very accurate. For example, a recent check** of the inflow from the drainage areas shows that two different methods of using the unit hydrograph method for converting rainfall into discharge (the difference consisted in taking 6-hour rather than 12-hour time intervals) resulted in discharges which differed by five to ten thousand cfs for a period of half a day.

The authors of this report feel that the inaccuracies in calculating the 1948 flood in the Ohio may well stem in large part from inaccuracies in the local inflows. The final results obtained for the 1945 flood were good, but they were

*This may also have been a factor in affecting the accuracy in the Ohio River, where the maximum rate of increase of stage was $H_t \doteq 0.9$ ft./hr., which is even larger than the maximum of 0.75 ft./hr. in Kentucky Reservoir.

**Made while this report was being written.

brought about by somewhat radical changes in cross section area and resistance coefficients. If it is true that such changes were made to compensate for errors in inflow rather than for inaccuracies in coefficients, it would not be surprising that subsequent calculations for another flood based on such coefficients would prove to be inaccurate. At the same time, it should also be pointed out that there are still other sources of error in the Ohio River problem, which were not investigated - above all the effect of a mesh width of 10 miles in the upper sections where the area coefficients varied considerably, and where rates of increase in stage were larger than those which occurred in Kentucky Reservoir.

b) Suggestions for improving the accuracy:

1. <u>Different interpolation schemes for cross section areas.</u>

In sec. 3 our method of dealing with cross section areas was explained in detail. For the purpose of the discussion here the main point is that average cross sections were obtained for each of the reaches (in the Ohio these were 60 - 90 miles long), these values were taken to hold at the centers of the reaches, and values at intermediate net points were obtained by linear interpolation from the center of one reach to the center of the next adjacent reach. The resulting cross section area at a given place then deviates often quite widely from the local value, especially at the gaging stations themselves which are usually placed at narrow parts of a river. However, our results show that the stages obtained by calculations based on such average cross section areas are accurate locally, but that the discharges obtained by multiplying the velocities obtained by us into the cross section areas used in our calculations can be quite wrong locally. This point has already been discussed in section 4 in connection with the Ohio River problem, and a satisfactory means to obtain correct discharges was given. Nevertheless, it would probably be better to revise our scheme of interpolation for cross section areas

and resistance coefficients in such a way as to bring the model of the river which the averaging process yields into somewhat closer agreement with the actual river. One way to do so would be to take the actual cross section areas at the ends of the reaches, then employ the total storage volumes for the reaches in such a way as to obtain an average cross section at the center of the reach, so that linear interpolation between it and the actual areas at the ends of the reach would yield the correct total storage volume in the reach. One could expect that correct values for the discharge at the gaging stations would then be obtained simply by multiplying the calculated velocity with the cross section area, which now would be the correct local value. We carried out this program partially for the Ohio River, but obtained bad results because we did not recalculate resistance coefficients, but used the same values as before. It is however, clear that the local value of G should be used at the gaging stations, since the average value used in the earlier calculation can depart widely from the local value, as we know from sec. 4. Time did not permit a revision of G in an appropriate fashion, but we would expect an improvement in the results if it were done. In addition, this new way of dealing with area and resistance coefficients would not be essentially harder to apply nor lead to more complications in coding than the former scheme.

2. Smoothing of cross section data.

In Kentucky Reservoir (cf. sec. 3 and the preceding sec. 6) we have seen that a new scheme for approximating derivatives was made necessary because of the rapid variation in the area coefficient along the reservoir. The data for cross section areas were supplied as averages for reaches ten miles in length (obtained from topographic maps), and these values differed by factors as large as two in adjacent reaches. Nevertheless, the data in this form were used by us, but a revision in the numerical methods was found to be necessary. It might have been better to have smoothed the average cross sections first, and then employed the smoothed

values as coefficients in the differential equations. Again, time did not permit a trial of such a method.

3. Cross section areas from topographic maps for the Upper Ohio River.

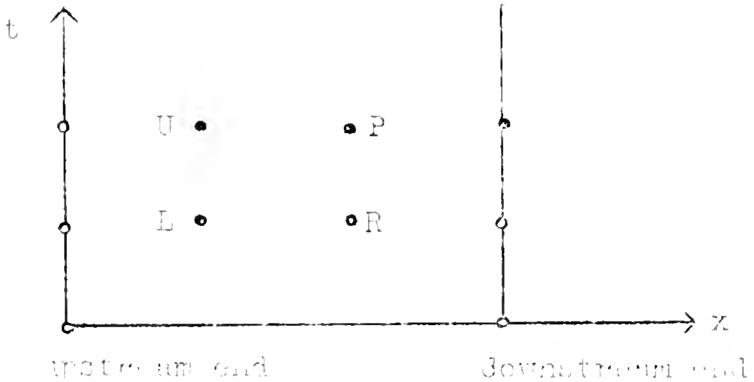
In view of the good results obtained in the junction problem and in Kentucky Reservoir, in which average cross section areas were obtained from topographic maps, it would probably be advisable to do the same thing for the Upper Ohio River. There, it will be recalled, the average cross sections were obtained from storage volumes in long reaches and these in turn resulted from balancing out inflows and outflows in actual floods. Thus the cross-section areas used by us contained errors due to faulty inflow data-and, as we have seen, the ungaged inflow, which can only be estimated with not too good accuracy, can be a quite considerable fraction of the total inflow. We would recommend in general the use of average cross sections obtained from maps since the extra labor involved would not be a large fraction of the total, and one would thus be sure of the accuracy of one of the most essential basic quantities.

4. Implicit schemes for numerical solution of the differential equations.

It has been stated repeatedly that our method of numerical solution of the differential equations, which proceeds by advancing the values of the stage and discharge from known values at time t to new and as yet unknown values at the time $t + \Delta t$, requires that the time step Δt must be kept very small (of the order of 9 minutes in our three cases) in comparison with the total time for which flow calculations would normally be made (of the order of weeks). The time required for making computations on a computing machine is inversely proportional to the time interval Δt. In our case, it did not matter too much, since the amount of time needed for calculation on the Univac was quite reasonable. Nevertheless, if much longer stretches of a river or river system were to be treated numerically,

this factor of machine time might become crucial. For this reason (and for others to be mentioned later), it would be worthwhile to study a radically different method of numerical solution in which the values of stage and velocity can be advanced in time steps of the order of hours instead of minutes, though at the expense of a more complicated computational scheme.

We proceed to discuss briefly one possible way to set up a numerical scheme of solution which might permit values for Δt of two or three hours. In the accompanying figure 7.1



a double row of net points in the x,t-plane is shown, and the object of the calculation is to advance the values of v and H using all of the values in both rows simultaneously. To this end the differential equations (2.1) with A_t replacing BH_t are written down for each rectangle, e.g. LRUP:

$$(7.1)\quad \frac{v_U + v_P - (v_L + v_R)}{2\Delta t} + \frac{(v_P^2 + v_R^2) - (v_U^2 + v_L^2)}{2\Delta x} + g\frac{(H_P + H_P) - (H_U + H_L)}{2\Delta x}$$

$$= -\left(\frac{G_U v_U^2 + G_P v_P^2 + G_L v_L^2 + G_R v_R^2}{4}\right) - \frac{1}{4}\sum_{U,P,L,R} \frac{Qv}{A}$$

and

$$(7.2) \quad \frac{-A_U v_U + A_P v_P - A_L v_L + A_R v_R}{2\Delta x} + \frac{A_U + A_P - (A_L + A_R)}{2\Delta t} = \frac{1}{2}(q_{UP} + q_{LR})$$

The unknowns in these equations are v_U, v_P, H_U, and H_P, i.e. two more than there are equations. However, if one boundary condition is given at each of the upstream and downstream ends (discharge, say, or a rating curve), the number of equations and the number of unknowns would be the same. In Fig. 7.1, for example, there would be eight values to be fixed at the four upper net points; there would be six equations of the type (7.1) and (7.2) and these together with two values at the boundaries (or two relations there), would yield a system of equations the same in number as the number of unknowns.* However, in marked contrast to the situation in the schemes used by us so far, the present system of equations would be a system of nonlinear simultaneous equations which could be solvable practically only by methods of successive approximation. Probably the best method to use for solving the equations would be an iterative method, with initial trial values for the unknowns selected by linear extrapolation from the results at previous rows of net points. Since we know that v and H do not change much for times as great as 2 or 3 hours, it seems quite possible that such a method would yield good approximations to the exact solution when time steps of these magnitudes are used.

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*It is the implicit character of this scheme that differentiates it from the previous schemes. The fact that all of the values at all net points for a time t are used simultaneously in determining the values for $t + \Delta t$ is the reason why the slope of the characteristics no longer dictates the maximum size permissible for Δt.

The implicit scheme would have still another advantage over former schemes which might increase the accuracy of the approximation. The advantage is that the implicit scheme provides automatically a means to ensure that the continuity condition holds in the large, i.e. that inflows and outflows would balance not only at each net point, but over-all, if such a computation were to be made. The methods used by us hitherto do not always check well for a whole river - in fact, in the Ohio fairly large discrepancies are found upon making a check for the whole river (This is understandable since the computed discharges have to be adjusted as described at the end of section 4). The same remark would also apply to the law of conservation of momentum, though with less force since we find this law to be well satisfied in the large when our methods are used.

Such implicit schemes have been proposed before for similar problems, but as far as we know, they have never been used on a large scale. It is also of interest to observe that H. A. Thomas (The Hydraulics of Flood Movements in Rivers, Carnegie Institute of Technology. 1937) proposed a variety of schemes for numerical solution of river wave problems, some of which would probably diverge, but also among them is a scheme essentially the same as the implicit scheme proposed here.

5. Simplified differential equations.

It might be possible to simplify the calculations somewhat by simplifying the differential equations, either through the omission of terms felt to be unimportant, or by developments and expansions of various kinds which yield simpler equations.

For example, the term vv_x in the dynamical equation could probably be omitted without causing much error, and also the term qv/A (the contribution of the inflow to the momentum). However, as long as one operates numerically with a system of hyperbolic equations (or even parabolic equations) the savings from simplifications in the differential equations are not

very great since the great bulk of the work consists in the processing of the basic data, and coding it. Still, it is by no means impossible that further studies could lead to formulations in which less elaborate calculating equipment than the Univac would suffice, at least for some types of problems. For example, a flood in a long river for which the inflows occur mainly in the head-waters so that the flood propagates essentially only in the downstream direction, with little or no backwater effects, very likely could be treated by simpler methods than those used by us in this report.

§8. Model Studies Compared with Numerical Studies Using Digital Computers

In the preceding sections some remarks have already been made concerning the relation between the methods explained here for the computation of flood waves in rivers through use of the basic differential equations as compared with the method of using physical models of a river or river system. These remarks will be amplified somewhat in this section.

A brief description of the procedures used in making model studies should first be given. Such models - at least those models of the Mississippi system which are now at the Waterways Experiment Station at Jackson, Mississippi - are built on a rather large scale, covering acres of ground in fact. They are built with a very much exaggerated vertical scale compared with the horizontal scale. Models are made of concrete with the channel being built up accurately from topographical surveys; this is a costly and time consuming feature. The inflows must be fed into the channel by rather complicated machines (pumps) which can reproduce any given discharges as functions of time. However, because of cost these are not placed at all of the main tributaries. Instead, the inflows over a considerable stretch of the river are lumped together to form a composite hydrograph; this hydrograph is reproduced by a machine, and the resulting discharge is fed into the river at a number of points in various proportions. The actual inflows are thus not put in accurately at the points where they enter the river, but rather, a certain average inflow is put in. (It is, however, probably a reasonable average.)

The river stages at the main gaging stations are then recorded electronically and the records can be seen in a special house containing recorders for each of the gaging stations. Air conditioning is necessary (just as with the

UNIVAC computer) because of the large amount of electronic equipment that is involved. All of this equipment is quite costly, and its operation requires a considerable staff.

It has already been mentioned that one of the main physical features governing the flow in a river, namely the roughness of the river bed which gives the flow its turbulent character, cannot be scaled properly for a model. Instead it is necessary to introduce obstacles in the bed of the model in order to reproduce the floods actually observed. At Jackson, these obstacles take the form of small brass knobs screwed into the bed of the model, or of wire screen, the latter placed as a rule in shallower parts of the stream. The fact is that the resistance force is, together with gravity, the biggest force conditioning the flow in the stream and the fact that this element of the problem cannot be scaled is a good reason for calling such models analogue computing machines rather than true scale models.

It has been stated above that the process of numerical computation of floods is analogous in all respects to the method of predicting floods using a model. The only difference is that we have made use of average cross section areas and resistance coefficients rather than actual local values as is done in the models*. It is the requirement of reproducing highly accurate cross sections that constitutes one of the costly features in building models. The experience reported above for the case of the junction problem and for Kentucky Reservoir shows that it is not necessary to use actual cross sections but rather that it suffices to use average cross sections over 10-20 mile stretches. Even in the Ohio River, where averages for reaches of 60 to 96 miles in length were

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* On the basis of our experience using average cross sections it is even indicated that models might also be built using average cross sections rather than actual cross sections, with the expectation of getting results having sufficient accuracy, and presumably at smaller cost. This in fact is an idea which was advocated long ago by the well-known hydraulics engineer H. A. Thomas.

used, the fact that the results were not as good as in the other two cases was quite likely less due to the use of averages over such long stretches than it was to the fact that the basic data did not furnish sufficiently accurate averages.

The next vital element involved both in the model and in the method of numerical computation concerns correct determination of the resistance factor. In models this is done in a completely empirical way in the manner indicated above. In making our numerical studies we could at least begin with a first reasonable estimate for this quantity based on analysis of actual observations. And, in the case of the junction problem and Kentucky Reservoir, where the area and resistance coefficients were well-determined by good basic data, little or no further experimentation with the resistance coefficient was required to reproduce the flood. True, it was found necessary in the less accurately defined Ohio River problem to correct the resistance factor by making predictions for a relatively short period and comparing with an actual flood. This corresponds exactly to what must be done in a model when brass knobs or wire screen are added or taken away from various parts of the model in order to reproduce observed floods.

There are various ways in which the method of numerical computation is much more flexible than the method of using models. In the first place it is quite easy to vary cross section areas if it is thought desirable to improve the accuracy. In a model that cannot be done readily; it perhaps should be done in some instances especially when the topographical data which were used for constructing the model are very old and there are possibilities that the river channel had changed in the meantime. Another feature in which numerical methods are more flexible concerns the method of dealing with the local runoff and the flow from tributaries. In operating with models it was indicated above that the inflows are fed into the model at a relatively small number

of points, after composite unit hydrographs have been calculated, since the apparatus needed for this purpose is rather complicated. In making numerical studies very little additional effort is required to introduce the flows at any point that might be desired. Of course the method of finite differences requires that the flows be introduced in the intervals between the net points. This, however, means in the cases studied by us that flows are introduced at points only 10 miles apart. That is, in each 10 mile interval the flow either from a tributary entering that interval or from the local runoff is introduced.

It is sometimes argued that the method of using a model has an advantage over the method of numerical calculation since with the model it is possible to observe what happens, say, in case a dike breaks and a flow out of the river channel takes place. However, it would seem to us that the results of such observations would be illusory unless observations on floods of this kind were available for the purpose of making verification runs in the model; while on the other hand, if such data were available they could be used to carry out predictions by numerical methods. In fact, as we have seen, it would be possible to estimate reasonable values for the resistance coefficient without any flow data, and proceed by numerical calculation to study flows where no such data were available.

In the end, presumably, it is the matter of cost which is the dominant feature in any comparison of the two methods. So far only the limited experience of this one group is available for estimating costs of the numerical method. However, in three years a relatively small group has been able to carry out the work outlines above starting with no previous experience. The total amount spent for the purpose of research and development of numerical methods including the cost of using the UNIVAC was well under $200,000.00. With the experience gained now it should be possible to code the data for equivalent stretches of other rivers for much less than

this. As for the amount of machine time needed for actual computation, we have seen it to be quite small. For the junction problem, for example, a 16 day prediction required less than 3 hours. The rental fee for the UNIVAC from the Army Map Service in Washington is only $75.00 an hour; privately owned calculating equipment is usually rented at somewhat higher rates than this but even so this cost is not large. if a model is built, it and the equipment which goes with it must be serviced and maintained, one would think at rather high cost. Once the data for a river have been coded for a machine, it is stored permanently in a few rolls of magnetic tape and is ready at any moment to be used to solve any flow problem; codes would also be prepared in advance so that the special initial and boundary data, and the inflows, all of which differ from case to case, could be rapidly prepared for use in the machine.

Finally, it should be remarked once more that calculating equipment is constantly being improved, so that our experience based on the use of a UNIVAC is not at all final. In fact, the use of the IBM 704 would probably reduce the calculating time needed for the problems discussed in this report in the ratio of about 1 to 15. Thus even the calculating time for the most difficult case, the Ohio, would be reduced from hours (for flows lasting weeks) to a matter of minutes.

Thus the authors are of the opinion that numerical methods offer great advantages in comparison with model studies for the types of problems dealt with in this report - i.e. wave problems in long rivers and reservoirs. However, this should not be taken to imply any adverse criticism of model studies in general, not even of those made in the past for large rivers - after all, the kind of computing equipment, and knowledge of numerical analysis, that was used for the problems discussed in this report has been available only for

a few years. Only in cases like the present ones, in which a reliable mathematical formulation of the problems in a not too complicated way is possible, can model studies be dispensed with. On the other hand, model studies could hardly be dispensed with for such a problem as, for example, the determination of the flow characteristics through the gates and turbines tributaries of Kentucky Dam itself, since an adequate purely mathematical formulation of such a problem would be, if not downright impossible, then at least enormously complicated.

§9. Conversion of Rainfall Data into Runoff Data for the Ohio River

The progress of a flood wave may depend decisively on the amount of water which is contributed by the ungaged runoff of rainwater coming from the area drained by the river: that is, the flow from the main valley and from small tributaries which are not gaged. The simple arithmetical calculation which converts rainfall data into runoff data has been coded and tested on the UNIVAC. It took less than 10 minutes of machine time to compute the runoff data for the 5 reaches for a 10 day period. All of our flood predictions were made with runoff data that was hand-computed and supplied to us by the Corps of Engineers. We merely report here on the obvious fact that it is feasible to prepare the runoff data on the UNIVAC and indicate the method that was used.

The Ohio River Division of the Corps of Engineers supplied us with the data on rainfall over each of the 5 drainage areas which are associated with the 5 reaches into which the upper Ohio River is divided. That is, the rainfall amounts for each 12 hour period, from 6:00 P.M. on Feb. 25, 1945 to midnight of Mar. 6, were given together with the percentages of these which represent the amount of water (called excess rainfall) which flows into the river according to certain fixed unit hydrograph proportions. Our calculations were made for the period from Feb. 26, 1945 through Mar. 6, 1945. (There was no rain for 4 days prior to Feb. 26.)
We used equation (9.1) to convert the 12 hour rainfall data for a reach into runoff data for the reach at 6 hour intervals (since the basic interval into which we divide the UNIVAC computation is 6 hours - that is, we represented $q(x,t)$ as being a linear function of the time, t, for a period of 6 hours).

$$(9.1) \quad q(t) = \begin{cases} u_2 \cdot r(t-36) + u_4 \cdot r(t-24) + u_6 \cdot r(t-12) + u_8 \cdot r(t) \ , & \text{for } t = \text{6:00 A.M. or 6:00 P.M.} \\ u_1 \cdot r(t-42) + u_3 \cdot r(t-30) + u_5 \cdot r(t-18) + u_7 \cdot r(t-6) + u_9 \cdot r(t+6) \ , & \text{for } t = \text{noon or midnight} \end{cases}$$

where $q(t)$ is the amount of overbank inflow for the reach expressed in units of one thousand cubic feet per second, while $r(y)$ is the amount of excess rainfall in inches that fell over the drainage area during the 12 hour period preceding y, i.e. from $(y-12)$ until y; and $u_1, u_2, \ldots, u_9$ (see Table 9.1) are the 6 hour instantaneous hydrograph coefficients for the drainage area. Note that only the rainfall of the preceding 2 1/2 days is permitted to contribute to the runoff

Reach / u	Wheeling St. Marys	St. Marys Pomeroy	Pomeroy Huntington	Huntington Maysville	Maysville Cincinnati
u_1	11,172	34,557	30,727	42,328	13,969
u_2	14,457	44,721	39,764	54,778	18,078
u_3	16,429	50,820	45,187	62,247	20,543
u_4	18,400	56,918	50,609	69,717	23,008
u_5	20,536	63,525	56,483	77,809	25,679
u_6	17,743	54,885	48,801	67,227	22.186
u_7	12,650	39,131	34,794	47,930	15,818
u_8	10,514	32,525	28,919	39,838	13,147
u_9	7,229	22,361	19,882	27,389	9,039

The inflow data that was used in our Ohio River flood calculations was read off at 6 hour intervals from curves of inflow prepared by the engineers with the use of 12 hour hydrograph coefficients. We compare the values of inflow obtained by us through the use of 6 hour hydrograph coefficients in equation (9.1) with the data supplied by the engineers; Fig. 9.1 is for the reach Wheeling to St. Marys and Fig. 9.2 is for the reach St. Marys to Pomeroy. It is seen that, as is to be expected, the 6 hour coefficients produced slightly higher crests and lower troughs than was the case for the 12 hour coefficients (since the peaks didn't occur at the ends of the 12 hour periods).

In particular, we noted that the short dry period which occurred on March 5, coincides with a brief drop in stage in the measured hydrograph at the stations between Wheeling and Cincinnati on that day. Our calculated hydrographs do not have as pronounced a dip and we attribute this fact in part to our having used the 12 hourly smoothed runoff data which do not have as pronounced a dry period on March 5 as actually occurred.

COMPARISON OF RUNOFF INFLOWS OBTAINED WITH
6 HOUR AND 12 HOUR COEFFICIENTS
(WHEELING — ST. MARYS)

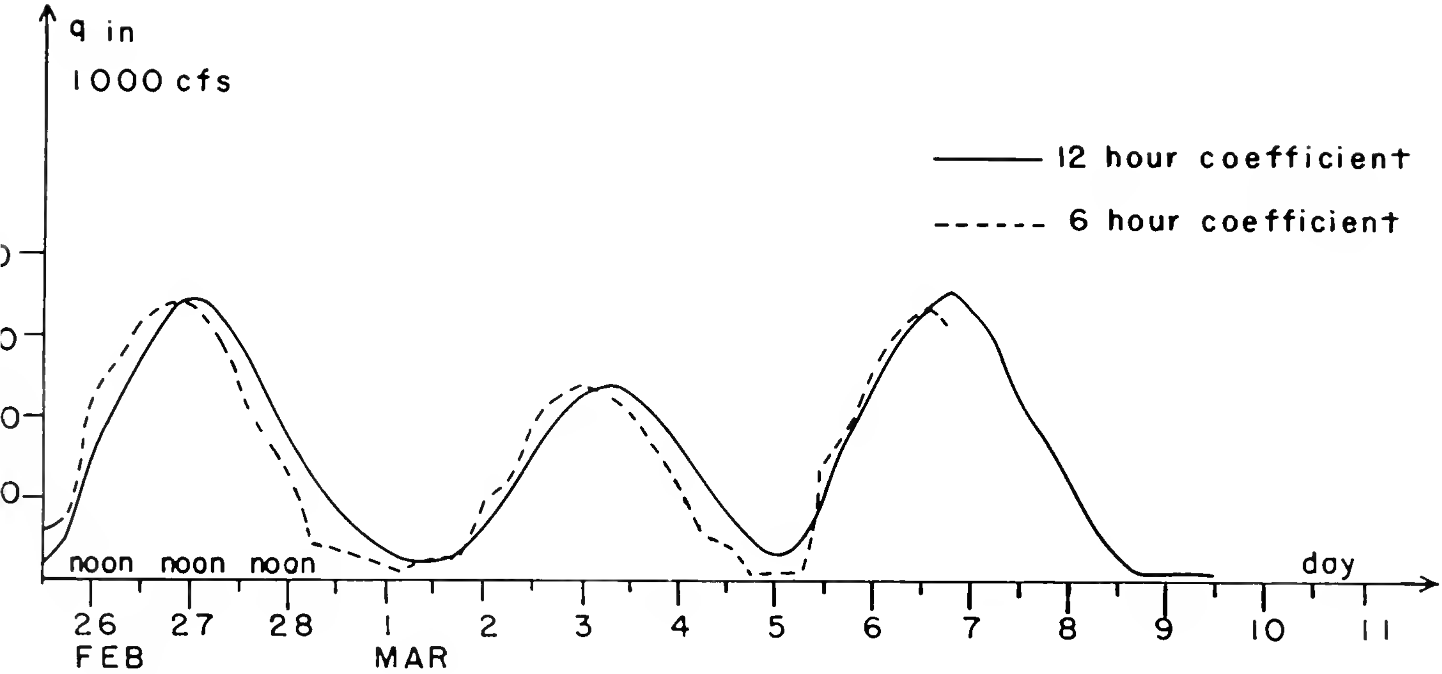

Fig. 9.1

COMPARISON OF RUNOFF INFLOWS OBTAINED
WITH 6 HOUR AND 12 HOUR COEFFICIENTS

(ST. MARYS—POMEROY)

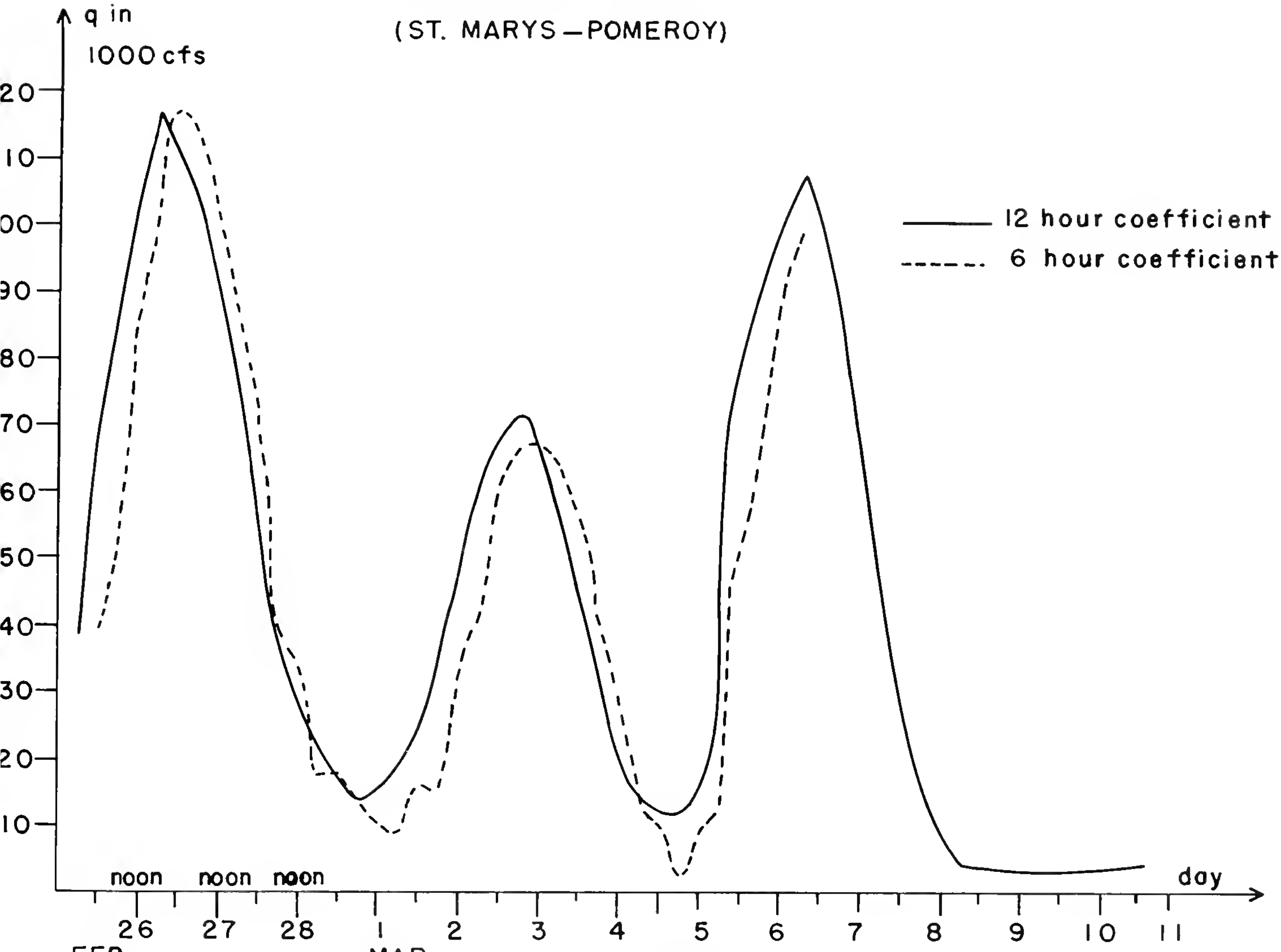

Fig. 9.2